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Prof. Dr. Günay ÖZTÜRK

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CONTENTS

CHAPTER 1

CHARACTERİZATİONS OF K-TYPE DARBOUX SLANT HELİCES IN $\mathbb{R}_1^4$

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1.INTRODUCTION

The theory of curves has been the subject of numerous areas' and geometry's recent investigations. Examining various geometric properties in Euclidean space using the theory of curves, which is the basis of differential geometry, and the Serret-Frenet frame of a curve, is a method of dealing with many geometric structures. Thus, Euclidean geometry has added many concepts to itself by walking on the theory of curves. These are topics such as the Bertrand curve, involute-evolute curve, helix and slant helix etc. In fact, the theory of curves and obtaining various geometric models of a curve using the Serret-Frenet frame have been used not only in geometry (and therefore mathematics) but also in different fields such as physics, biology and engineering. On the other hand, the theory of curves and the analysis of various geometric models in different spaces, namely Lorentz and Minkowski spaces, using the Serret-Frenet frame of a curve and presenting new theories are the methods followed by differential geometry. In particular, the fact that the curves in the structures of Lorentz and Minkowski spaces and their Serret-Frenet frame contain more special situations than Euclidean space, allowing for various situations and more study areas. From this point of view, geometry offers a tremendous field of study.

Many researchers who specifically research the literature well can see that helix and slant helix issues have been proven by building many theories in Lorentz and Minkowski spaces. Although the subject of slant helices has been discussed in the literature, it has not been studied in a comprehensive manner. This lack of detailed treatment forms the main motivation of the present study. Slant helices were first introduced as a generalization of classical helices. $k-$ and $(k,m)-$type slant helices have been added to the literature in recent years, after the slant helices. Akgün [1] introduced the concept of new kind Frenet curves within the frame of Minkowski space. Ali and Önder [2] provided characterizations for rectifying space-like curves in Minkowski space-time. Bulut and Eker [3] developed the theory of $k-$ and $(k,m)-$type slant helices with respect to the Lorentz-Darboux frame. The geometry of special helices on equiform differential geometry of timelike curves was discussed by Bulut [4]. Bulut and Tartık [5] studied *(k,m)*-type slant helices by employing the parallel transport frame in Euclidean 4-space. The structure of slant helices relative to the Ed-frame in Minkowski 4-space was analyzed by Bulut [6]. Bulut and Bektaş [7] investigated special helices arising from the equiform differential geometry of spacelike curves in Minkowski space-time. Bulut [8] formulated non-linear differential equations based on the Darboux vector approach.

Camcı, İlarslan, Kula, and Hacısalihoğlu [9] examined harmonic curvatures and generalized helices in four-dimensional Euclidean space. Düldül [10] analyzed vector fields and planes in $\mathbb{R}^4$ that act analogously to the Darboux vector. Izumiya and Takeuchi [11] introduced novel types of special curves and associated developable surfaces. Keleş, Yüksel Perktaş, and Kılıç [12] explored the properties of biharmonic curves in LP-Sasakian manifolds. O'Neill [13] provided a comprehensive treatment of semi-Riemannian geometry with significant applications to the theory of relativity. Öztürk, Gürpınar, and Arslan [14] proposed a novel characterization of curves within Euclidean 4-space. Walrave [15] conducted an extensive study on curves and surfaces in Minkowski space through his doctoral dissertation. Williams and Stein [16] discussed a triple product of vectors within four-dimensional space. Yaylı, Gök, and Hacısalihoğlu [17] introduced extended rectifying curves as a new class of modified Darboux vectors. Yılmaz and Bektaş [18] investigated $(k,m)-$ type slant helices associated with partially null and pseudo null curves in Minkowski space. Yılmaz and Bektaş [19] further contributed to the understanding of slant helices of *(k,m)*-type in four-dimensional Euclidean space.

In this paper, we explore the properties of k- type slant helices in 4-dimensional Minkowski space $\mathbb{R}_1^4$, B_1 is time-like vector, focusing on the case where the playing role Darboux vector. This vector plays role as the Darboux vector in 4-dimensional Minkowski space $\mathbb{R}_1^4$, and using in the geometric characterization of slant helices. We also derive non-linear equations which provided several characterizations of k- type slant helices of curves in which the vector B_1 is time-like, in accordance with the Frenet frame fields associated. Furthermore, we offer an analysis of the curvature structure of these slant helices, aiming to deepen the understanding of their behavior within Lorentzian geometry. The results presented here contribute to the broader understanding of curve theory in Minkowski 4-space and open the door to further exploration of k- type slant helices under various geometric constraints. Additionally, numerical examples are provided to illustrate the applications of the theoretical results. The figures are obtained using the computed values of the curvatures. In Examples 1 and 2, unit speed B_1 is time-like curves are considered, and their curvatures and related invariants are explicitly computed. Using these values, the corresponding nonlinear differential equations with the given coefficients are solved. Approximate solution curves for Examples 1 and 2 are then drawn and visualized in four-dimensional space using the R programming language. In both examples, the objective is to demonstrate the applicability of the theoretical findings, particularly the solutions of the nonlinear differential equations associated with the curvature functions, to specific B_1 is time-like curves and their visualization in higher-dimensional spaces.

2.GENERAL PROPERTIES OF METHOD

In this part, we introduced some basic definition for Minkowski space. The Minkowski space $\mathbb{R}_1^4$ is the standard vector space equipped with an indefinite flat metric given by:

$$\langle , \rangle = da_1^2 + da_2^2 + da_3^2 - da_4^2,$$

where (a_1, a_2, a_3, a_4) is a rectangular coordinate system of $\mathbb{R}_1^4$. A vector u in $\mathbb{R}_1^4$ is called a space-like, time-like or null (light-like), if holds $\langle u, u\rangle > 0$, $\langle u, u\rangle < 0$ or $\langle u, u\rangle = 0$ $(u \neq 0)$ respectively. The norm of the vector u is given by $\| u \| = \sqrt{|\langle u, u\rangle|}$. if $\langle u, w\rangle = 0$, then two vectors, u and w are said to be orthogonal. If all of the velocity vectors $\alpha'(s)$ associated with an arbitrary curve $\alpha: I \to \mathbb{R}_1^4$ are space-like, time-like, or null, then the curve can be respectively space-like, time-like, or null. Let $\{T, N, B_1, B_2\}$ be the moving Frenet frame along the curve $\alpha(s)$ in $\mathbb{R}_1^4$. Then the vector fields T, N, B_1, B_2 are the tangent, the principal normal, the first binormal and the second binormal vector fields respectively. Let α be a space-like curve in $\mathbb{R}_1^4$, parametrised by the arc length function of s. The following Lemma 1 follow for the time-like curve α in [15].

Lemma 1. Let the vector N be space-like and B_1 be time-like. In this instance, $\alpha(s)$ is a time-like curve with Frenet equations

$$\begin{aligned} T' &= k_1 N \\ N' &= -k_1 T + k_2 B_1 \\ B_1' &= k_2 N + k_3 B_2 \\ B_2' &= k_3 B_1 \end{aligned} \tag{1}$$

for which there is only one Frenet frame T, N, B_1, B_2 where T, N, B_1 and B_2 are mutually orthogonal vectors satisfying the equations $<T,T>=<N,N>=<B_2,B_2>=1$ and $<B_1,B_1>=-1$. Recall that the functions $k_1 = k_1(s)$, $k_2 = k_2(s)$ and $k_3 = k_3(s)$ are called the first, second and third curvature of the time-like curve $\alpha(s)$ respectively and we will assume throughout this work that all the three curvatures satisfy $k_i(s) \neq 0$, $1 \leq i \leq 3$ in [15].

Let us set that $V_1 = T, V_2 = N, V_3 = B_1, V_4 = B_2$.

Definition 1. A time-like curve $\gamma(s)$ parametrized by arc-length s with Frenet frame $\{V_1, V_2, V_3, V_4\}$ (or with Darboux vector fields $\{D_1, D_2, D_3, D_4\}$) in Minkowski space $\mathbb{R}_1^4$ is called a k- type slant helix (or is called a k-type Darboux slant helix) for $k \in \{1,2,3,4\}$ provided that the subsequent holds true for

$$\langle V_k, \mathbf{U} \rangle = constant \ \ (or \ \langle D_k, \mathbf{U} \rangle = constant)$$

and there is a non-zero fixed vector $\mathbf{U} \in \mathbb{R}_1^4$ in [18]. In this section, we define Darboux vector fields by using Lemma along a regular curve in $\mathbb{R}_1^4$. By using $k-$type slant helices we obtain some non-linear first order differential equations. Let γ be a unit speed curve as given in the Lemma and $\{T, N, B_1, B_2\}$ be the Frenet frame along the curve γ where the curvatures k_1, k_2, k_3 are non-zero everywhere. Now we can define following vector fields along γ:

$$\begin{aligned} D_1 &= B_2 \\ D_2 &= k_2 T + k_1 B_1 \\ D_3 &= k_3 N + k_2 B_2 \\ D_4 &= -T \end{aligned} \tag{2}$$

where $\{D_1, D_2, D_3, D_4\}$ is linearly independent along γ in [1].

Definition 2. Let $\gamma(s)$ be an $\mathbb{R}_1^4$ curve. If the position vector of $\gamma(s)$ is always in the orthogonal complement of its principal normal vector field, it is referred to as a rectifying curve; if it is always in the orthogonal complement of its first binormal vector field, it is referred to as an osculating curve of the first sort in [14].

2.1. Non-Linear Differential Equations of K-type slant helices

In this section, we calculate non-linear equation according to Frenet vectors and curvatures of curves in Minkowski space $\mathbb{R}_1^4$.

Theorem 1. Assume that γ is a curve that the vector N be space-like and B_1 be time-like with a Frenet frame of $\{T, N, B_1, B_2\}$ in $\mathbb{R}_1^4$. There is the non-linear equation,

$$\frac{1}{k_3^2} y'^2 - y^2 - \mu = 0$$

if the curve γ is a 1-type slant helix and a 1-type Darboux slant helix in $\mathbb{R}_1^4$ where $y = \frac{k_1}{k_2}$ and $\mu = \frac{K^2}{c_1^2} - 1$ because c_1, K are constants.

Proof. In $\mathbb{R}_1^4$, let $\gamma = \gamma(s)$ be a 1-type slant helix and a 1-type Darboux slant helix. If $\mathbf{U}$ is a definite direction that is not zero, then

$$\langle T, \mathbf{U} \rangle = c_1 \tag{3}$$

is a constant along the curve $\gamma = \gamma(s)$. By setting (2) in (3), we find

$$\langle D_4, \mathbf{U} \rangle = -c_1.$$

Differentiating (3) with respect to s, we get

$$\langle T', \mathbf{U} \rangle = 0. \tag{4}$$

Using equation (1), we find the following equation:

$$k_1 \langle N, \mathbf{U} \rangle = 0$$

and $k_1 \neq 0$, we obtain

$$\langle N, \mathbf{U} \rangle = 0. \tag{5}$$

Differentiating (5) with respect to s and using equations (2) and (3), we have

$$-k_1 c_1 + k_2 \langle B_1, \mathbf{U} \rangle = 0$$

it follows that

$$\langle B_1, \mathbf{U} \rangle = \frac{k_1}{k_2} c_1. \qquad (6)$$

Differentiating (6) with respect to s and using (1) and (5), we have

$$\langle B_2, \mathbf{U} \rangle = \frac{1}{k_3} \left(\frac{k_1}{k_2}\right)' c_1. \qquad (7)$$

$\mathbf{U}$ a constant field, we can write

$$\mathbf{U} = u_1 T + u_2 N + u_3 B_1 + u_4 B_2 \qquad (8)$$

and for K is a constant

$$\langle \mathbf{U}, \mathbf{U} \rangle = K^2 = constant. \qquad (9)$$

This suggests that the subspace spanned by $\{T, B_1, B_2\}$ contains the unit vector $\mathbf{U}$. Using equation (1), we obtain the components of $\mathbf{U}$ as follows:

$$\begin{aligned} u_1 &= \langle T, \mathbf{U} \rangle = c_1 \\ u_2 &= \langle N, \mathbf{U} \rangle = 0 \\ u_3 &= \langle B_1, \mathbf{U} \rangle = -\frac{k_1}{k_2} c_1 \\ u_4 &= \langle B_2, \mathbf{U} \rangle = \frac{c_1}{k_3} \left(\frac{k_1}{k_2}\right)' c_1. \end{aligned}$$

Thus, we have

$$\mathbf{U} = c_1 T - \frac{k_1}{k_2} c_1 B_1 + \frac{c_1}{k_3} \left(\frac{k_1}{k_2}\right)' B_2. \qquad (10)$$

If we standard scalar product both sides of equation (10) by $\mathbf{U}$, we find

$$K^2 = c_1^2 - c_1^2 \left(\frac{k_1}{k_2}\right)^2 + \frac{c_1^2}{k_3^2} \left(\left(\frac{k_1}{k_2}\right)'\right)^2. \qquad (11)$$

If we divide both sides of equation (11) by c_1^2, we have $\frac{K^2}{c_1^2} - 1 = -\left(\frac{k_1}{k_2}\right)^2 + \frac{1}{k_3^2} \left(\left(\frac{k_1}{k_2}\right)'\right)^2$. If $\mu = \frac{K^2}{c_1^2} - 1 = constant$ is taken, we obtain $\frac{1}{k_3^2} \left(\left(\frac{k_1}{k_2}\right)'\right)^2 - \left(\frac{k_1}{k_2}\right)^2 - \mu = 0$. If $y = \frac{k_1}{k_2}$ is taken, we obtain the following non-linear equation,

$$\frac{1}{k_3^2} y'^2 - y^2 - \mu = 0. \qquad (12)$$

The proof is completed.

Theorem 2. Let γ is a curve that the vector N be space-like and B_1 be time-like in $\mathbb{R}_1^4$ with Frenet frame $\{T, N, B_1, B_2\}$. If the curve γ is a 2-type slant helix and 2-type Darboux slant helix in $\mathbb{R}_1^4$, then we obtain the following non-linear equation,

$$y'^2 - \frac{2k_2}{k_1} \varepsilon' y' + \left(k_3^2 + \frac{k_2^2 k_3^2}{k_1^2}\right) y^2 + \left[\frac{k_2^2}{k_1^2} \varepsilon'^2 - \mu \frac{k_3^2}{k_1^2} \varepsilon'^2\right] = 0$$

where $y = \frac{k_1}{k_1^2+k_2^2}$ and $\mu = \frac{K^2}{c_2^2} - 1$, ε are constants.

Proof. In $\mathbb{R}_1^4$, let $\gamma = \gamma(s)$ be a 2-type slant helix and a 2-type Darboux slant helix. If $\mathbf{U}$ is a definite direction that is not zero, then

$$\langle N, \mathbf{U} \rangle = c_2 \tag{13}$$

is a constant along the curve $\gamma = \gamma(s)$. Differentiating (13) with respect to s, we get

$$\langle N', \mathbf{U} \rangle = 0. \tag{14}$$

From $\gamma = \gamma(s)$ is a 2-type Darboux slant helix, we can write

$$\langle D_2, \mathbf{U} \rangle = c_5. \tag{15}$$

By setting (1) in (14), we find the following equation:

$$-k_1\langle T, \mathbf{U} \rangle + k_2\langle B_1, \mathbf{U} \rangle = 0 \tag{16}$$

and substituting (15) to (1), we obtain as below:

$$k_2\langle T, \mathbf{U} \rangle + k_1\langle B_1, \mathbf{U} \rangle = c_5. \tag{17}$$

By setting (16) in (17), we find

$$\langle T, \mathbf{U} \rangle = \left(\frac{k_2}{k_1^2 + k_2^2}\right) c_5$$

it follows that

$$\langle B_1, \mathbf{U} \rangle = \left(\frac{k_1}{k_1^2+k_2^2}\right) c_5. \tag{18}$$

Differentiating (18) with respect to s, we get $\langle B_2, \mathbf{U} \rangle = -\left(\frac{k_2}{k_3}\right) c_2 + \frac{1}{k_3}\left(\frac{k_1}{k_1^2+k_2^2}\right)' c_5$. $\mathbf{U}$ a constant field, we can write

$$\mathbf{U} = u_1 T + u_2 N + u_3 B_1 + u_4 B_2 \tag{19}$$

and for K is a constant $\langle \mathbf{U}, \mathbf{U} \rangle = K^2 = constant.$ This suggests that the subspace spanned by $\{T, B_1, B_2\}$ contains the unit vector $\mathbf{U}$. Using equation (1), we obtain the components of $\mathbf{U}$ as follows:

$$\begin{aligned} u_1 &= \langle T, \mathbf{U} \rangle = \left(\frac{k_2}{k_1^2+k_2^2}\right) c_5 \\ u_2 &= \langle N, \mathbf{U} \rangle = c_2 \\ u_3 &= \langle B_1, \mathbf{U} \rangle = -\left(\frac{k_1}{k_1^2+k_2^2}\right) c_5 \\ u_4 &= \langle B_2, \mathbf{U} \rangle = -\left(\frac{k_2}{k_3}\right) c_2 + \frac{1}{k_3}\left(\frac{k_1}{k_1^2+k_2^2}\right)' c_5. \end{aligned} \tag{20}$$

By setting (20) in (19), it can be written as $\mathbf{U} = \left(\frac{k_2}{k_1^2+k_2^2}\right) c_5 T + c_2 N - \left(\frac{k_1}{k_1^2+k_2^2}\right) c_5 B_1 + \left(-\left(\frac{k_2}{k_3}\right) c_2 + \frac{1}{k_3}\left(\frac{k_1}{k_1^2+k_2^2}\right)' c_5\right) B_2$. If we standard scalar product both sides of the last equation by $\mathbf{U}$ and $c_5 = \frac{k_1}{\varepsilon'} c_2$ where $\varepsilon' = \left(\frac{k_2}{k_1^2+k_2^2}\right)'$, we find

$$K^2 = \frac{k_1^2}{\varepsilon'^2}\left(\frac{k_1}{k_1^2+k_2^2}\right)^2 c_2^2 + c_2^2 + \left(\frac{k_1^2}{k_1^2+k_2^2}\right)^2 \frac{c_2^2}{\varepsilon'^2}\left(-\frac{k_2}{k_3} + \frac{k_1}{k_3\varepsilon'\left(\frac{k_1}{k_1^2+k_2^2}\right)'}\right)^2 c_2^2. \tag{21}$$

If we divide both sides of equation (21) by c_2^2 and $\mu = \frac{K^2}{c_2^2} - 1$ are taken, we obtain the following non-linear equation,

$$y'^2 - \frac{2k_2}{k_1}\varepsilon' y' + \left(k_3^2 + \frac{k_2^2 k_3^2}{k_1^2}\right) y^2 + \left[\frac{k_2^2}{k_1^2}\varepsilon'^2 - \mu\frac{k_3^2}{k_1^2}\varepsilon'^2\right] = 0.$$

The proof is completed.

Corollary 1. Let γ is a curve that the vector N be space-like and B_1 be time-like in $\mathbb{R}_1^4$ with non-zero curvatures k_1, k_2 and k_3. If the curve γ is a 2-type slant helix and 2-type Darboux slant helix in $\mathbb{R}_1^4$, then the following holds

$$\frac{c_2}{c_5} = \frac{1}{k_1}\left(\frac{k_2}{k_1^2+k_2^2}\right)' = constant \neq 0.$$

Theorem 3. Let γ is a curve that the vector N be space-like and B_1 be time-like in $\mathbb{R}_1^4$ with Frenet frame $\{T, N, B_1, B_2\}$. If the curve γ is a 3-type slant helix and 3-type Darboux slant helix in $\mathbb{R}_1^4$, then we obtain the following non-linear equation

$$y'^2 + \frac{2k_2}{k_3}\lambda' y' + \left(k_1^2 + \frac{k_1^2 k_2^2}{k_3^2}\right) y^2 + \left[\frac{k_2^2}{k_3^2}\lambda'^2 - \mu\frac{k_1^2}{k_3^2}\lambda'^2\right] = 0$$

where $y = \frac{k_3}{k_2^2 - k_3^2}$ and $\mu = \frac{K^3}{c_3^2} + 1$, λ' are constants.

Proof. In $\mathbb{R}_1^4$, let $\gamma = \gamma(s)$ be a 3-type slant helix and a 3-type Darboux slant helix. If $\mathbf{U}$ is a definite direction that is not zero, then

$$\langle B_1, \mathbf{U}\rangle = c_3 \tag{22}$$

is a constant along the curve $\gamma = \gamma(s)$. Differentiating (22) with respect to s, we get

$$\langle B_1', \mathbf{U}\rangle = 0. \tag{23}$$

From $\gamma = \gamma(s)$ is a 3-type Darboux slant helix, we can write

$$\langle D_3, \mathbf{U}\rangle = c_6. \tag{24}$$

By setting (1) in (23), we find the following equation:

$$k_2\langle N, \mathbf{U}\rangle + k_3\langle B_2, \mathbf{U}\rangle = 0 \tag{25}$$

and substituting (24) to (2), we obtain as below:

$$k_3\langle N, \mathbf{U}\rangle + k_2\langle B_2, \mathbf{U}\rangle = c_6. \tag{26}$$

By setting (25) in (26), we find

$$\langle N, \mathbf{U}\rangle = \frac{-c_6k_3}{k_2^2-k_3^2} \tag{27}$$

it follows that

$$\langle B_2, \mathbf{U}\rangle = \frac{c_6k_2}{k_2^2-k_3^2}. \tag{28}$$

Differentiating (27) with respect to s, we get $\langle T, \mathbf{U}\rangle = \frac{k_2}{k_1}c_3 + \frac{1}{k_1}\left(\frac{k_3}{k_2^2-k_3^2}\right)' c_6$.

Differentiating (28) with respect to s, we get $c_6 = \frac{k_3}{\left(\frac{k_2}{k_2^2-k_3^2}\right)'}c_3$. $\mathbf{U}$ a constant field, we can write

$\mathbf{U} = u_1T + u_2N + u_3B_1 + u_4B_2$

and for K is a constant $\langle \mathbf{U}, \mathbf{U}\rangle = K^2 = constant$. This suggests that the subspace spanned by $\{T, B_1, B_2\}$ contains the unit vector $\mathbf{U}$. Using equation (1), we obtain the components of $\mathbf{U}$ as follows:

$$\mathbf{U} = \left(\frac{k_2}{k_1}c_3 + \frac{1}{k_1}\left(\frac{k_3}{k_2^2-k_3^2}\right)' c_6\right)T + \left(\frac{-c_6k_3}{k_2^2-k_3^2}\right)N - c_3B_1 + \left(\frac{c_6k_2}{k_2^2-k_3^2}\right)B_2. \tag{29}$$

If we standard scalar product both sides of equation (29) by $\mathbf{U}$ and $c_6 = \frac{k_3}{\lambda'}c_3$ is written, we find

$$K^2 = \frac{1}{k_1^2}\left(k_2 + \frac{k_3}{\lambda'}\left(\frac{k_3}{k_2^2-k_3^2}\right)'\right)^2 c_3^2 + \left(\frac{1}{\lambda'}\left(\frac{k_3^2}{k_2^2-k_3^2}\right)\right)^2 c_3^2 - c_3^2 + \left(\frac{k_3}{\lambda'}\left(\frac{k_2}{k_2^2-k_3^2}\right)\right)^2 c_3^2,$$

where $\lambda' = \left(\frac{k_2}{k_2^2-k_3^2}\right)'$. If we divide both sides of last equation by c_3^2 and $\mu = \frac{K^2}{c_3^2} + 1$, $y = \left(\frac{k_3}{k_2^2-k_3^2}\right)'$ is taken, we obtain non-linear equation.

The proof is completed.

Corollary 2. Let γ is a curve that the vector N be space-like and B_1 be time-like in $\mathbb{R}_1^4$ with non-zero curvatures k_1, k_2 and k_3. If the curve γ is a 3-type slant helix and 3-type Darboux slant helix in $\mathbb{R}_1^4$, then the following holds

$$\frac{c_3}{c_6} = \frac{1}{k_3}\left(\frac{k_2}{k_2^2 - k_3^2}\right)' = constant \neq 0.$$

Theorem 4. Let γ is a curve that the vector N be space-like and B_1 be time-like in $\mathbb{R}_1^4$ with Frenet frame $\{T, N, B_1, B_2\}$. If the curve γ is a 4-type slant helix and 4-type Darboux slant helix in $\mathbb{R}_1^4$, then we obtain the following non-linear equation,

$$\frac{1}{k_1^2} y'^2 - y^2 - \mu = 0$$

where $y = \frac{k_3}{k_2}$ and $\mu = \frac{K^2}{c_4^2} - 1$, because c_4, K are constants.

Proof. In $\mathbb{R}_1^4$, let $\gamma = \gamma(s)$ be a 4-type slant helix and a 4-type Darboux slant helix. If **U** is a definite direction that is not zero, then

$$\langle B_2, \mathbf{U}\rangle = c_4 \qquad (30)$$

is a constant along the curve $\gamma = \gamma(s)$. From $\gamma = \gamma(s)$ is a 4-type Darboux slant helix, we find $\langle D_1, \mathbf{U}\rangle = c_4$. Differentiating (30) with respect to s, we get $\langle B_2', \mathbf{U}\rangle = 0$. **U** a constant field, we can write $\mathbf{U} = u_1 T + u_2 N + u_3 B_1 + u_4 B_2$ and for K is a constant $\langle \mathbf{U}, \mathbf{U}\rangle = K^2 = constant$.

This suggests that the subspace spanned by $\{T, B_1, B_2\}$ contains the unit vector **U**. Using equation (1), we obtain the components of **U** as follows:

$$\mathbf{U} = \frac{1}{k_1}\frac{k_3{}'}{k_2} c_4 T - \frac{k_3}{k_2} c_4 N + c_4 B_2. \qquad (31)$$

If we standard scalar product both sides of equation (31) by **U**, we find

$$K^2 = c_4^2 - c_4^2\left(\frac{k_1}{k_2}\right)^2 + \frac{c_4^2}{k_3^2}\left(\left(\frac{k_1}{k_2}\right)'\right)^2. \qquad (32)$$

If we divide both sides of equation (32) by c_4^2, we have $\frac{K^2}{c_4^2} - 1 = -\left(\frac{k_3}{k_2}\right)^2 + \frac{1}{k_1^2}\left(\left(\frac{k_3}{k_2}\right)'\right)^2$. If $\mu = \frac{K^2}{c_4^2} - 1$ is taken, we obtain

$$\frac{1}{k_1^2}\left(\left(\frac{k_3}{k_2}\right)'\right)^2 - \left(\frac{k_3}{k_2}\right)^2 - \mu = 0. \tag{33}$$

If $y = \frac{k_3}{k_2}$ is taken, we obtain the non-linear equation and so the proof is completed.
A numerical example is provided for theorem 4 and its figure are as follows are demonstrated using found values for k_1 and μ.

3.APPLICATIONS

A numerical example is provided for theorem 1 and its figure are as follows are demonstrated using found values for k_3 and μ.
Example 1. Let γ be a unit speed time-like curve in R_1^4 , given by the equation
$y(s) = \left(\frac{1}{2}\cos\left(\sqrt{2s}\right), \frac{1}{2}\sin\left(\sqrt{2s}\right), \cosh\left(s\right), \sinh\left(s\right)\right)$. We easily obtain the curvatures and μ as follows: $k_1(s) = \frac{\sqrt{5}}{2}, k_1(s) = \frac{3}{2\sqrt{5}}, k_1(s) = \frac{2}{\sqrt{5}}, \mu = -\frac{25}{9}$ and so, the solution of according to the non-linear equation (12) with coefficient μ, we get

$$\frac{k_1(s)}{k_2(s)} = \frac{5\tan\left(\frac{2(-s+i)\sqrt{5}}{s}\right)}{\sqrt[3]{-1-\tan\left(\frac{2(-s+i)\sqrt{5}}{s}\right)^2}}$$

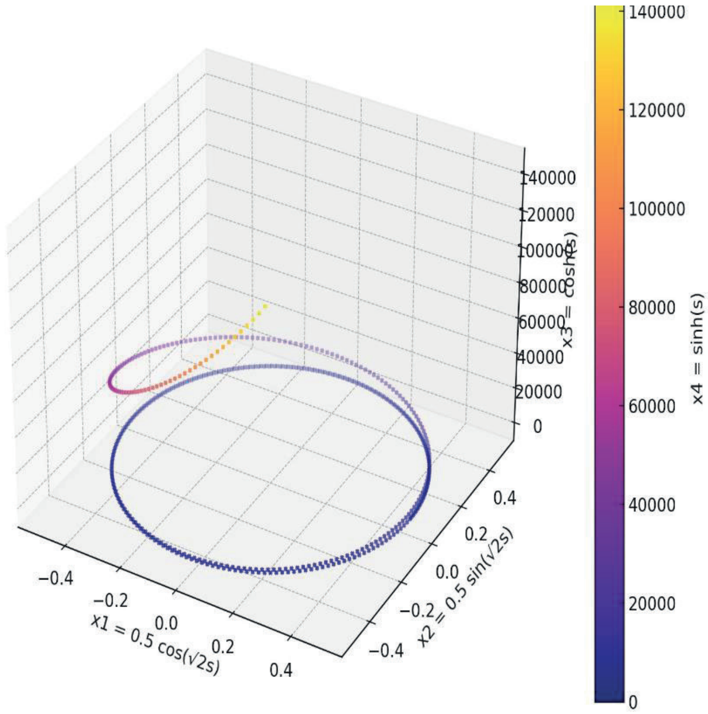

Figure 1. Approximate solution curves of Example 1.

Example 2. Let γ be a unit speed time-like curve in $\mathbb{R}_1^4$ with the equation $\gamma(s) = \left(\frac{1}{3}cos(\sqrt{3}s), \frac{1}{3}sin(\sqrt{3}s), cosh(s), sinh(s)\right)$. We easily obtain the curvatures and μ as follows:
$k_1(s) = \frac{\sqrt{13}}{3},\ k_2(s) = \frac{14}{3\sqrt{26}},\ k_3(s) = \frac{3\sqrt{2}}{\sqrt{13}},\ \mu = -\frac{81}{49}$ and the solution according to the non-linear equation (33), yield the following equation:

$$\frac{k_3(s)}{k_2(s)} = \frac{9tan\left(\frac{(-s+i)\sqrt{13}}{3}\right)}{7\sqrt{-1-tan\left(\frac{(-s+i)\sqrt{13}}{3}\right)^2}}.$$

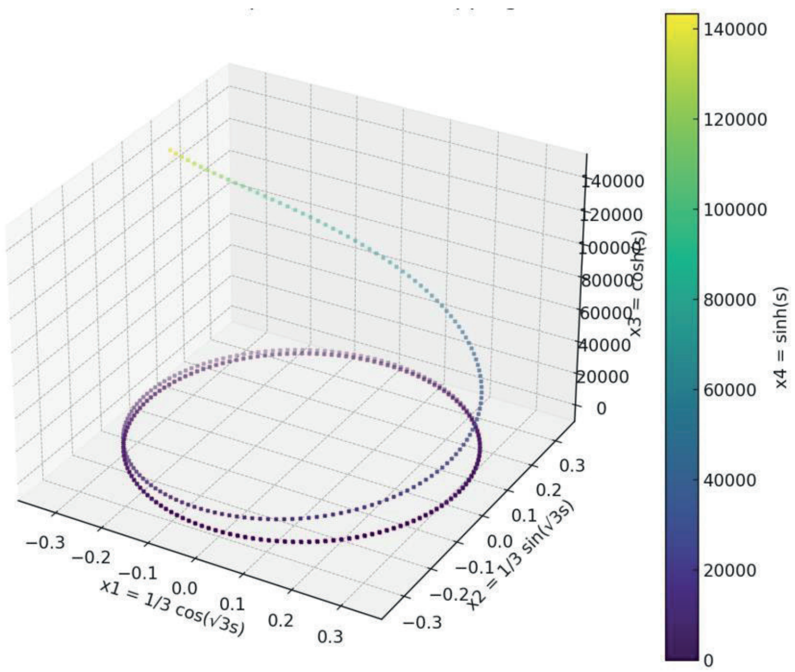

Figure 2. Approximate solution curves of Example 2.

4.CONCLUSIONS

The slant helices in Minkowski 4-space $\mathbb{R}_1^4$ exhibit rich geometric structures that differ from their Euclidean counterparts due to the indefinite nature of the metric. The distinction between space-like and time-like components in the Frenet frame leads to unique behaviors and differential relationships. In particular, the existence of distinct nonlinear equations characterizing each k-type slant helix reveals that these curves are governed by intricate geometric constraints. Moreover, the utilization of the Darboux vector frame adds further geometric insight into the internal symmetries of these curves. These results may have applications in relativity theory, where the Minkowski structure naturally appears, and in modeling trajectories constrained by Lorentzian geometry.

In this paper, we have investigated the geometry of k-type slant helices in Minkowski 4-space $\mathbb{R}_1^4$ under the assumption that the binormal vector B_1 is time-like. By employing the Frenet and Darboux frames, we derived several nonlinear differential equations corresponding to

$k = 1, 2, 3, 4$. These equations provide necessary conditions for a curve to be classified as a k-type slant helix or Darboux slant helix.

We explore the relationship between k-type slant helices and their corresponding k-type Darboux slant helices, as well as the nonlinear differential equations of the slant helices that depend on the various geometric and analytic variables involved. This work lays the groundwork for future studies that may examine conditions under which the constants defining the k-type slant helices and the k-type Darboux slant helices coincide. Our results not only generalize classical slant helix concepts into the Lorentzian context but also offer new insights into the interplay between curvature functions and geometric invariants in higher-dimensional semi-Riemannian spaces. The inclusion of numerical examples and visualizations further confirms the theoretical findings. Future studies may extend this approach to partially null or pseudo null curves, or explore similar structures in other signature spaces, or extending these results to higher-dimensional pseudo-Riemannian manifolds.

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CHAPTER 2

ON THE UNIQUENESS OF T HE SOLUTION OF THE INVERSE PROBLEM OF SCATTERING THEORY FOR STURM-LIOUVILLE OPERATOR WITH DISCONTINUOUS COEFFICIENT

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1.Introduction

Let us consider the inverse scattering problem for the differential equation

$$-\psi''(x)+q(x)\psi(x)=\lambda^2\rho(x)\psi(x), \qquad 0<x<+\infty, \tag{1}$$

subject to the boundary condition

$$-\left(\alpha_1\psi(0)-\alpha_2\psi'(0)\right)=\lambda^2\left(\beta_1\psi(0)-\beta_2\psi'(0)\right), \tag{2}$$

where λ is the spectral parameter.

The potential function $q(x)$ is assumed to be real-valued and to satisfy the integrability condition

$$\int_0^{+\infty}(1+x)\left|q(x)\right|dx<\infty. \tag{3}$$

Furthermore, $\rho(x)$ is a positive piecewise-constant function with a finite number of points of discontinuity. The constants α_i and β_i $(i=1,2)$ are real numbers satisfying

$$\gamma:=\alpha_1\beta_2-\alpha_2\beta_1>0.$$

The aim of this work is to investigate both the direct and inverse scattering problems on the half-line $[0,+\infty)$ for the boundary value problem (1)- (3)

2. Related Works

In the case $\rho(x)\equiv 1$, the inverse scattering problem for equation (1)} with boundary conditions not containing the spectral parameter was completely solved by Marchenko (Marchenko1955, Marchenko1986), Levitan (Levitan1975, Levitan1987), Aktosun (Aktosun2004), as well as Aktosun and Weder (Aktosun-Weder2006).

The version of the problem with discontinuous coefficients was investigated by Gasymov (Gasymov1977) and Darwish (Darwish1993). In these works, the solution of the inverse scattering problem on the half-line $[0,+\infty)$ by means of transformation operators was reduced to the solution of two inverse problems on the intervals $[0,a]$ and $[a,+\infty)$.

In the case $\rho(x)\neq 1$, the inverse scattering problem was studied by Guseinov and Pashaev (Guseinov-Pashaev2002), who employed a new non-triangular representation of the Jost solution corresponding to equation (1). It was shown that the discontinuity of the function $\rho(x)$ strongly influences both the structure of the Jost solution and the main equation of the inverse problem.

We note that similar phenomena do not arise for systems of Dirac equations with discontinuous coefficients, as shown by Mamedov and Çöl (Mamedov-Col2008).

The uniqueness of the solution of the inverse problem and its geophysical applications for equation (1) in the case $q(x) \equiv 0$ were investigated by Tikhonov (Tikhonov1949) and Alimov (Alimov1976).

The inverse problem for a wave equation with a piecewise constant coefficient was studied by Lavrent'ev (Lavrentev1992).

When $\rho(x) \equiv 1$ and the spectral parameter appears in the boundary conditions of equation (1), the inverse problem on the half-line was considered by Pocheykina and Fedotova (Pocheykina1972) using the spectral function approach, by Yurko (Yurko2000a, Yurko2000b, Yurko2002) using the Weyl function, and by Mamedov (Mamedov2003,Mamedov2009) based on scattering data.

Boundary value problems with spectral parameter-dependent boundary conditions arise in a wide range of physical and applied problems, including heat conduction problems studied by Cohen (Cohen1966) and wave equations investigated by Yurko (Yurko2000a, Yurko2000b).

Spectral analysis of Sturm--Liouville problems on the half-line was examined by Fulton (Fulton1977). Physical applications of problems with linear dependence on the spectral parameter in the boundary conditions on a finite interval were discussed by Fulton (Fulton1981).

On a finite interval, inverse spectral problems for Sturm--Liouville operators with linear or nonlinear dependence on the spectral parameter in the boundary conditions were studied by Chernozhukova and Freiling (ChernozhukovaFreiling2009), Chugunova (Chugunova2001), Rundell and Sacks (Rundell-Sacks2004)}, Guliyev (Guliyev2005), and Mamedov and Çetinkaya (Mamedov-Cetinkaya2014, Mamedov-Cetinkaya2013).

3. **Scattering Data and the Inverse Problem**

For the boundary value problem (1)- (3) the following results are obtained:

a) the scattering data corresponding to the boundary value problem (1)- (3) are defined;

b) the main equation of the inverse problem is derived;

c) the uniqueness of the solution of the main equation is proved;

d) the uniqueness of the solution of the inverse problem is established;

a) a Levinson-type formula is obtained.

For simplicity, we assume that in equation (1) the function $\rho(x)$ has a single point of discontinuity, namely

$$\rho(x)=\begin{cases}\alpha^2, & 0\le x<a\\ 1, & x>0\end{cases} \tag{1.4}$$

where $0<\alpha\ne 1$.

The function

$$f_0(x,\lambda)=\frac{1}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)e^{i\lambda\mu^+(x)}+\frac{1}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)e^{i\lambda\mu^-(x)}, \tag{5}$$

where

$$\mu^{\pm}(x)=\pm x\sqrt{\rho(x)}+a\left(1\mp\sqrt{\rho(x)}\right),$$

is the Jost solution of equation (1) for $q(x)\equiv 0$.

It is known from (Guseinov-Pashaev1998) that for all λ in the closed upper half-plane, equation (1) has a unique Jost solution $f(x,\lambda)$ satisfying

$$\lim_{x\to+\infty} f(x,\lambda)e^{-i\lambda x}=1, \tag{6}$$

and admitting the representation

$$f(x,\lambda)=f_0(x,\lambda)+\int_{\mu^+(x)}^{+\infty}K(x,t)e^{i\lambda t}\,dt, \tag{7}$$

where the kernel $K(x,t)$ satisfies

$$\int_{\mu^+(x)}^{+\infty}|K(x,t)|\,dt\le C\exp\left(\int_x^{+\infty}t\,|q(t)|\,dt\right),\quad 0<C=\text{const.} \tag{8}$$

Moreover, the kernel $K(x,t)$ possesses the properties

$$\frac{d}{dx}K(x,\mu^+(x))=-\frac{1}{4\sqrt{\rho(x)}}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)q(x), \tag{9}$$

$$\frac{d}{dx}\left\{K(x,\mu^{-}(x)+0)-K(x,\mu^{-}(x)-0)\right\}=\frac{1}{4\sqrt{\rho(x)}}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)q(x). \qquad (10)$$

If $q(x)$ is differentiable, then $K(x,t)$ satisfies almost everywhere

$$\rho(x)\frac{\partial^2 K}{\partial t^2}-\frac{\partial^2 K}{\partial x^2}+q(x)K=0, \quad 0<x<+\infty, \quad t>\mu^{+}(x). \qquad (11)$$

For real $\lambda \neq 0$, the functions $f(x,\lambda)$ and $\overline{f(x,\lambda)}$ form a fundamental system of solutions of the equation (1) and their Wronskian satisfies

$$W\{f(x,\lambda),\overline{f(x,\lambda)}\}=2i\lambda,$$

where $W\{f,g\}=f'g-fg'$.

Let $w(x,\lambda)$ be the solution of equation (1) satisfying

$$w(0,\lambda)=\alpha_2+\beta_2\lambda^2, \qquad w'(0,\lambda)=\alpha_1+\beta_1\lambda^2. \qquad (12)$$

Then for all real $\lambda \neq 0$ the identity

$$\frac{2i\lambda w(x,\lambda)}{(\alpha_2+\beta_2\lambda^2)f'(0,\lambda)-(\alpha_1+\beta_1\lambda^2)f(0,\lambda)}=\overline{f(x,\lambda)}-S(\lambda)f(x,\lambda) \qquad (13)$$

holds, where

$$S(\lambda)=\frac{(\alpha_2+\beta_2\lambda^2)\overline{f'(0,\lambda)}-(\alpha_1+\beta_1\lambda^2)\overline{f(0,\lambda)}}{(\alpha_2+\beta_2\lambda^2)f'(0,\lambda)-(\alpha_1+\beta_1\lambda^2)f(0,\lambda)}. \qquad (14)$$

Moreover,

$$S(\lambda)=\overline{S(-\lambda)}=[S(-\lambda)]^{-1}.$$

The function $S(\lambda)$ is called *the scattering function* of the boundary value problem (1)-(3).

The function $\varphi(\lambda)$ has only finitely many zeros in the half-plane $\operatorname{Im}\lambda>0$, all of which are simple and lie on the imaginary axis.

Define

$$m_k^{-2} = \int_0^{+\infty} \rho(x) \left| f(x, i\lambda_k) \right|^2 dx + \frac{1}{\gamma} \left| \beta_2 f'(0, i\lambda_k) - \beta_1 f(0, i\lambda_k) \right|^2$$
$$= \frac{1}{2i\lambda_k \gamma} \varphi'(i\lambda_k) \left[\beta_2 f'(0, i\lambda_k) - \beta_1 f(0, i\lambda_k) \right], \qquad k = 1, 2, \ldots, n.$$

These quantities are called *the normalizing numbers* of the boundary value problem (1)-(3).

The collection

$$\{S(\lambda), -\infty < \lambda < +\infty; \lambda_k; m_k, k = 1, 2, \ldots, n\}$$

is called *the scattering data* of the boundary value problem (1)-(3).

The inverse scattering problem consists in recovering the potential $q(x)$ from the scattering data.

For each fixed $x \geq 0$, the kernel $K(x, y)$ satisfies the integral equation

$$F(x, y) + \int_{\mu^+(x)}^{+\infty} K(x, t) F_0(t + y) dt + K(x, y) + \frac{1 - \sqrt{\rho(x)}}{1 + \sqrt{\rho(x)}} K(x, 2a - y) = 0, \qquad (15)$$

where

$$F_0(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[S_0(\lambda) - S(\lambda) \right] e^{-i\lambda x} d\lambda + \sum_{k=1}^{n} m_k^2 e^{-\lambda_k x}, \qquad (16)$$

$$F(x, y) = \frac{1}{2} \left(1 + \frac{1}{\sqrt{\rho(x)}} \right) F_0(y + \mu^+(x)) + \frac{1}{2} \left(1 - \frac{1}{\sqrt{\rho(x)}} \right) F_0(y + \mu^-(x)). \qquad (17)$$

Equation (15) is called *the main equation* of the inverse scattering problem. Due to the discontinuity of $\rho(x)$, it differs essentially from the classical Marchenko equation and is therefore referred to as *the modified Marchenko equation*.

Using the main equation, it is shown that the scattering function $S(\lambda)$ is continuous for all real λ, and the main equation admits a unique solution $K(x, \cdot) \in L_1(\mu^+(x), \infty)$ for each $x \geq 0$.

Consequently, the scattering data uniquely determine the boundary value problem (1)-(3).

4. Levinson-Type Formula

We now present a Levinson-type formula which establishes a relation between the increment of the argument of the scattering function $S(\lambda)$ and the number of eigenvalues of the boundary value problem (1)-(3).

Theorem (Levinson-Type Formula)

The following identity holds:

$$\frac{\ln S(+0)-\ln S(+\infty)}{2\pi i}+C(\beta_2)-\frac{1-S(0)}{4}=n,$$

where n *denotes the number of eigenvalues of the boundary value problem (1)-(3) and the constant* $C(\beta_2)$ *is defined by*

$$C(\beta_2)=\begin{cases}\frac{3}{2}, & \beta_2\neq 0,\\ 1, & \beta_2=0.\end{cases}$$

5. An Application

In classical quantum mechanics, the stationary state of a system consisting of two particles with masses m_1 and m_2 and total energy E is described by the wave function Ψ, which satisfies the Schrödinger equation

$$-\frac{h^2}{2M}\Delta\Psi+V(x)\Psi=E\Psi, \tag{18}$$

where h is Planck's constant,

$$M=\frac{m_1m_2}{m_1+m_2}$$

is the reduced mass, $V(x)$ is the interaction potential, and $x=|\vec{x}|$ denotes the distance between the two particles.

Since the potential $V(x)$ depends only on the distance $|\vec{x}\,|$, separation of variables in

The equation (18) yields *the separation equation*

$$\Psi(\vec{x}) = x^{-1} u_{\ell}(E,x) Y_{\ell}^{m}(\theta,\varphi),$$

where $Y_{\ell}^{m}(\theta,\varphi)$ are the spherical harmonics. The radial function $u_{\ell}(E,x)$ satisfies the differential equation

$$-\frac{h^2}{2M}\left(u_{\ell''} - \frac{\ell(\ell+1)}{x^2}u_{\ell}\right) + V(x)u_{\ell} = Eu_{\ell}, \qquad (19$$

together with the boundary condition

$$u_{\ell}(E,0) = 0.$$

Introducing the notations

$$q(x) = \frac{2M}{h^2}V(x), \qquad \lambda^2 = \frac{2ME}{h^2}, \qquad (20)$$

The equation (19) can be rewritten as the boundary value problem

$$-u_{\ell''} + q(x)u_{\ell} + \frac{\ell(\ell+1)}{x^2}u_{\ell} = \lambda^2 u_{\ell}, \qquad 0 < x < \infty, \qquad (21)$$

subject to

$$u_{\ell}(0) = 0. \qquad (22)$$

The solutions of the boundary value problem (20-22) that remain bounded as $x \to \infty$ are called *the radial wave functions*.

We assume that the potential satisfies

$$\int_0^{\infty} x|\,q(x)\,|dx < \infty. \qquad (23)$$

Under this condition, it follows from the results of the previous sections that for $\ell = 0$ the problem (20)-(22) admits bounded solutions $u_0(x)$ corresponding to $\lambda^2 > 0$ and to discrete eigenvalues $\lambda = i\lambda_k$ $(k = 1,\ldots,n)$. Moreover, as $x \to \infty$,

$$u_0(x) = e^{-i\lambda x} - S(\lambda)e^{i\lambda x} + o(1), \qquad 0 < \lambda^2 < \infty, \qquad (24)$$

and

$$u_0(i\lambda_k, x) = m_k e^{-\lambda_k x}(1 + o(1)), \qquad k = 1, \ldots, n. \tag{25}$$

Thus, the scattering data

$$\{S(\lambda), -\infty < \lambda < \infty; \lambda_k, m_k, k = 1, \ldots, n\} \tag{26}$$

provide a complete description of the asymptotic behavior of all radial wave functions $u_0(x)$. A similar description holds for $\ell \neq 0$.

In terms of the scattering data, the potential $V(x)$ is uniquely recovered. In particular, if $K(x, y)$ denotes the kernel of the transformation operator, then the potential $q(x)$ is given by

$$q(x) = -\frac{1}{2}\frac{d}{dx}K(x, x).$$

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CHAPTER 3

SUMMABILITY FACTOR RELATIONS BETWEEN ABSOLUTE q- CESÀRO METHODS

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1. INTRODUCTION

From antiquity to present, sequences and series have constituted one of the central notions in mathematics. The idea of convergence of series, which is a well-established concept, traces its origins back to ancient times. Before the concept of convergence was precisely defined, mathematicians often encountered paradoxical results and unsolvable inconsistencies when attempting to assign sums to infinite series using arbitrary operations. Some of these contradictions were clarified through Gauss's binomial theorem. Later, Cauchy, by formalizing the concept of convergence for sequences and series, provided a systematic foundation and offered a new perspective. This development marked the beginning of the modern understanding of convergence and divergence. Nevertheless, Cauchy's formulation naturally gave rise to a further question: *Is it possible to assign a sum to a divergent series?* The affirmative answer to this question came through the extension of convergence to broader notions, giving rise to the theory of summability.

To illustrate the idea, let us consider the following example, which highlights the essence of summability in a simple manner. For $|z| < 1$, it is well known that

$$1 + z + z^2 + \cdots = \frac{1}{1-z} \,.$$

Substituting $z = -1$ into this identity, Euler obtained

$$1 - 1 + 1 - 1 + \cdots = \frac{1}{2}. \tag{1}$$

In the framework of Cauchy convergence, there is a problem, since the series

$$\sum_{k=0}^{\infty} (-1)^k$$

diverges. However, when the sequence of partial sums (b_m) is transformed by the first-order Cesàro mean, one obtains

$$t_n = \frac{1}{n+1} \sum_{k=0}^{n} b_k = \frac{1}{2} + \frac{1}{4(n+1)} [1 + (-1)^n].$$

Here, it is clear that (t_n) converges to $1/2$, $n \to \infty$. Hence, the sum of the non-convergent series given by (1) is calculated as $1/2$ using the Cesàro summability method. This example shows that divergent series can be summed, if the method is changed, so it is very important in terms of summability theory. This classical example demonstrates that divergent series can indeed be summed under suitable transformations, hence underlining the significance of summability theory.

Summability theory continues to hold an important role across various branches of mathematics and the applied sciences. It is widely utilized in engineering disciplines, applied analysis, functional analysis, Fourier analysis, and related areas. In this regard, the literature on summability theory has evolved in two principal directions: first, through the construction, study, and characterization of new summability methods and their associated sequence spaces generated by classical matrices such as Hausdorff, Hölder, Fibonacci, Cesàro, Nörlund, and Euler; and second, through the development of new absolute summability methods and sequence spaces obtained via absolute summability methods from different perspectives. More recently, absolute summability methods have gained increasing importance, leading to the development of novel sequence spaces that provide fresh insights and broaden the scope of the field. As a result, summability theory continues to make substantial contributions to contemporary mathematical research and remains a dynamic and influential area of study. Further examples and related discussions can be found in (see (Bor, 1993; Çınar and Et, 2020; Erdem, 2024; Ilkhan, 2020; Sulaiman, 1992; Yaying et al., 2025; Yaying et al, 2021)).

On the other hand, one of the key concepts addressed in this study is the q-analogue of a mathematical expression, which involves generalizing the expression by introducing a parameter q. As $q \to 1$, the q- analogue naturally reduces to its classical method. Although the origins of q-calculus can be traced back to the work of Euler, it has become a more vibrant and actively researched field in recent years. q-calculus has attracted attention due to its wide range of applications in mathematics, physics, and engineering. It finds extensive use in various branches of mathematics, including approximation theory, combinatorics, quantum algebra, special functions, operator theory, hypergeometric functions, and beyond. The q-Cesáro matrix $C^q = (c^q_{nv})$, which is one of the basic concepts of this study, has recently been defined by Aktuğlu and Bekar (2011) as follows:

$$c^q_{nv} = \begin{cases} \dfrac{q^v}{[n+1]_q}, & 0 \le v \le n \\ 0, & v > n \end{cases}$$

where $[n]_q$ is the q-analogue of a non-negative number n and identified by

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q}, & q \epsilon R^+ - \{1\} \\ n, & q = 1. \end{cases}$$

To highlight the distinction between q-analogue summability and its classical counterparts, consider the sequence

$$x_n = (-1)^n.$$

This sequence is not classically convergent, but becomes summable for $0 < q < 1$ under a suitable q-Cesàro summability method (see Figure 1 for $q = 0.5, s = 1, n = 100$). Beyond these theoretical aspects, applications to Fourier series and q-difference equations reveal the practical importance of q- Cesàro methods**.** They provide a refined tool for capturing subtle forms of convergence, including weak or statistical types. Furthermore, in engineering contexts such as signal processing and data compression, it is often desirable to suppress noise or smooth irregular fluctuations. While traditional Cesàro means apply uniform averaging, this may not suffice for highly variable data. In contrast, the q-Cesàro framework includes an adjustable q-parameter that allows the researcher to balance smoothing and the preservation of local details, offering greater versatility. In future work, the potential use of q-Cesàro summability in adaptive signal filtering, noise reduction and compression algorithms deserves further attention. Figure 1 already suggests that when $q < 1$, q-Cesàro smoothing outperforms the classical scheme, especially in the presence of rapid oscillations.

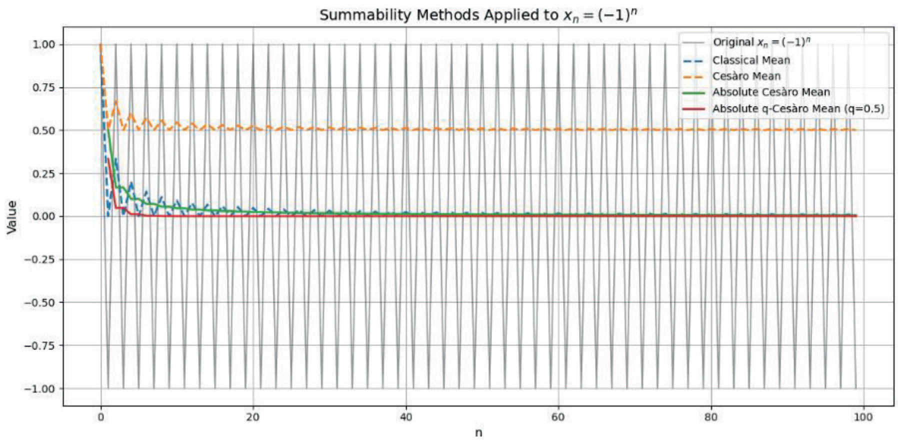

Figure 1

Nevertheless, some limitations should be acknowledged. Calculations based on q-calculus such as q-differences or q-summability operators are typically more complicated than classical formulations, and proofs involving q-Cesàro matrices often require considerable technical effort. Moreover, the parameter q is not universal; its choice crucially influences the outcome. Therefore, q must be selected carefully according to the nature of the application, rather than chosen arbitrarily.

Let $b = (b_j)$ be sequence of partial sum of the series $\sum u_i$, and let $\varphi = (\varphi_j)$ be any sequence of positive real numbers, $\mu = (\mu_j)$ be any bounded sequence of positive real numbers. Following Gökçe & Sarıgöl, (2018), the series $\sum u_i$ is said to be summable $|\Lambda, \varphi|(\mu)$, if

$$\sum_{j=1}^{\infty} \varphi_j^{\mu_n - 1} \left|\Lambda_j (b) - \Lambda_{j-1} (b)\right|^{\mu_n} < \infty.$$

The summability method $|\Lambda, \varphi|(\mu)$ is highly general and encompasses many well-known absolute summability methods as special cases, depending on the choice of the matrix Λ and the sequences φ and μ. For example, if one takes the triangle matrix T instead of Λ with $\mu_j = s$ for all j, the summability method $|\mathrm{T}, \varphi|_s$ is immediately obtained (Gökçe, 2022). Similarly, choosing the Euler matrix with $\mu_j = s$ for all j, yields the summability methods $|E^r, \varphi|_s$ satisfying the condition

$$\sum_{n=1}^{\infty} \theta_n^{s-1} \left| \sum_{v=1}^{n} \binom{n-1}{v-1} (1-r)^{n-v} r^v u_v \right|^s < \infty,$$

(Gökçe & Sarıgöl, 2020), if we decide on the weighted mean matrix instead of Λ, the summability method $|\Lambda, \theta_n|(\mu)$ is reduced to the $|\bar{N}, p_n, \theta_n|(\mu)$ satisfying the condition

$$\sum_{n=1}^{\infty} \theta_n^{\mu_n - 1} \left| \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} u_v \right|^{\mu_n},$$

(Gökçe & Sarıgöl, (2018)), if we decide on the Cesàro matrix instead of Λ with $\theta_n = n$ for all n, the method is reduced to $|C, \alpha, \beta|(\mu)$ and the set of all series summable by this method is given by:

$$\sum_{n=1}^{\infty} n^{\mu_n - 1} \left| \sum_{v=0}^{n} \left(\frac{A_{n-v}^{\lambda-1}}{A_n^{\lambda+\beta}} - \frac{A_{n-v-1}^{\lambda-1}}{A_{n-1}^{\lambda+\beta}} \right) A_v^{\beta} b_v \right|^{\mu_n} < \infty,$$

(Gökçe & Sarıgöl, (2019)). In addition to the last chosing above, if $\beta = 0$ and $\mu_n = s$ for all n is selected, the method $|C, \alpha|_s$ given by the condition

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| \sum_{v=0}^{n} \frac{A_{n-v}^{\lambda-1}}{A_n^{\lambda}} v u_v \right|^s < \infty$$

studied by Sarıgöl is obtained, (Sarıgöl, 2016).

Let $\lambda = (\lambda_n)$ be a sequence and X and Y be two summability methods. If $\sum \lambda_n u_n$ is summable Y whenever $\sum u_n$ is summable the summability method X, then λ is said to be a summability factor of type (X, Y), and we denote it by $\lambda \epsilon (X, Y)$. The problems of summability factors dealing with absolute q-Cesàro summability is the main subject of the study. Before moving on to the main theorems, let us recall the lemmas that will be used in the proofs:

Lemma 1. (Stieglitz, Tietz, 1977) Let $1 < s < \infty$. $\Lambda \in (l_s, l)$ if and only if

$$\sup \left\{ \sum_{j=0}^{\infty} \left| \sum_{n \in N} \lambda_{nj} \right|^{s^*} : N \subset \mathbb{N} \text{ finite} \right\} < \infty.$$

Lemma 2 (Sarıgöl, 2013) Let $\Lambda = (\lambda_{ji})$ be an infinite matrix with complex components, $\varrho = (\varrho_i)$ be a bounded sequence of positive numbers. If $W_\varrho\,[\Lambda\,] < \infty$ or $L_\varrho\,[\Lambda\,] < \infty$, then

$$(2m)^{-2} W_\varrho\,[\Lambda\,] \le L_\varrho\,[\Lambda\,] \le W_v\,[\Lambda\,],$$

where $m = \max\{1, 2^{M-1}\}, M = \sup_i \varrho_i$.

$$W_{\varrho}\,[\Lambda\,] \;= \sum_{i=0}^{\infty}\left(\sum_{j=0}|\lambda_{ji}|\right)^{\varrho_i}$$

and

$$L_{\varrho}\,[\Lambda\,] = \sup\left\{\sum_{i=0}^{\infty}\left|\sum_{j\in G}\lambda_{ji}\right|^{\varrho_i} : G\subset\mathbb{N}\text{ finite}\right\}.$$

Lemma 3. (Maddox, 1970) $\Lambda\in(\,l,l_s)$ if and only if

$$\sup_j\sum_{n=0}^{\infty}|\lambda_{nj}|^s<\infty,$$

where $1\;\le\;s\;<\;\infty$.

2. MAIN RESULTS

In this part of the section, we introduce the absolute q-Cesàro summability method which combines the notion of absolute summability with the transformation matrix generated by the q-Cesàro matrix. Subsequently, the theorems expressing the necessary and sufficient conditions for $\lambda\epsilon(|C^q,\varphi|_s,|C^p,\delta|)$ and $\lambda\epsilon(|C^p,\delta|,|C^q,\varphi|_s)$, which are the main problem of the study, will be stated and proven. To obtain this method, let us take the sequence $\sum u_i$ and its partial sums b_j. Then we get

$$\Lambda_{\mathrm{n}}(b)=\sum_{i=0}^{n}c_{ni}^{q}b_i=\sum_{j=0}^{n}u_j\sum_{i=j}^{n}\frac{q^i}{[n+1]_q}=\sum_{j=0}^{n}u_j\,(1-\frac{[j]_q}{[n+1]_q})$$

and so,

$$\Delta\Lambda_n(b)=\sum_{j=0}^{n}u_j\left(1-\frac{[j]_q}{[n+1]_q}\right)-\sum_{j=0}^{n-1}u_j\left(1-\frac{[j]_q}{[n]_q}\right)$$

$$=\sum_{j=1}^{n}\frac{q^n[j]_q}{[n]_q[n+1]_q}u_j,\quad n>0,$$

$$\Delta\Lambda_0(b)=u_0.$$

If

$$\sum_{n=0}^{\infty}\varphi_n^{s-1}\,|\,\Delta\Lambda_n(b)|^s<\infty,$$

it is said that the series $\sum u_i$ is summable by the method $|C^q,\varphi|_s$. Also, considering the transformation sequence (T_n), it can be written that the series $\sum u_i$ is summable by $|C^q,\;\varphi|_s$ if and only if $(T_n)\epsilon\;l_s$. Here

$$T_n=\varphi_n^{1/s^*}\sum_{j=1}^{n}\frac{q^n[j]_q}{[n]_q[n+1]_q}u_j,\quad n>0,$$

$$T_0 = \varphi_0^{1/s^*} u_0.$$

By making a few calculations, it can be seen that the inverse transformation of the transformation sequence (T_n) is as follows:

$$u_n = T_n \frac{[n+1]_q}{\varphi_n^{1/s^*} q^n} - T_{n-1} \frac{[n-1]_q}{\varphi_{n-1}^{1/s^*} q^{n-1}}, n > 0, \qquad (2)$$

$$u_0 = \varphi_0^{-1/s^*} T_0.$$

Throughout the article, it is noted that (φ_j) and (δ_j) represent any sequences of positive numbers, and also s^* indicates the conjugate of s that is $1/_s + 1/_{s^*} = 1$ for $s > 0$, $1/_{s^*} = 0$ for $s = 1$.

It can be easily seen that, in case of $q = 1$ and $\varphi_n = n$, the summability method $|C^q, \varphi|_s$ is reduced the well-known classical absolute Cesàro method $|C, 1|_s$, (Rhoades,1998).

For simplicity, throughout the rest of the article, it will be used that

$$[j]_p [j+1]_q - [j+1]_p [j]_q = \Delta_{pq}(j).$$

Theorem 1. Assume that $1 < s < \infty$. Then, $\lambda \epsilon (|C^q, \varphi|_s, |C^p, \delta|)$ if and only if

$$\sum_{j=0}^{\infty} \left(\left| \varphi_j^{-1/s^*} \frac{p^j [j+1]_q}{q^j [j+1]_p} \lambda_j \right| + \left| \frac{\sigma_{j+1}^{(p)}}{\varphi_j^{1/s^*} q^j} \left(\lambda_j \Delta_{pq}(j) + \Delta\lambda_j [j+1]_p [j]_q \right) \right| \right)^{s^*} < \infty$$

where

$$\sigma_{j+1}^{(p)} = \begin{cases} \dfrac{1}{j+1}, & p = 1 \\ \dfrac{p^{j+1}}{[j+1]_p}, & p < 1 \\ \dfrac{1}{[j+1]_p}, & p > 1. \end{cases}$$

Proof.

Assume that T_n and t_n are the transformation sequences of $|C^q, \varphi|_s$ and $|C^p, \delta|$ means of series $\sum u_n$and $\sum \lambda_n u_n$ respectively, that is

$$T_n = \varphi_n^{1/s^*} \sum_{j=1}^{n} \frac{q^n [j]_q}{[n]_q [n+1]_q} u_j, t_n = \sum_{j=1}^{n} \frac{p^n [j]_p}{[n]_p [n+1]_p} \lambda_j u_j.$$

Considering the inverse transformation of T_n, it is easily obtained that

$$t_n = \sum_{j=1}^{n} \frac{p^n [j]_p}{[n]_p [n+1]_p} \lambda_j u_j$$

$$= \sum_{j=1}^{n} \frac{p^n [j]_p \lambda_j}{[n]_p [n+1]_p} \left(T_j \frac{[j+1]_q}{\varphi_j^{1/s^*} q^j} - T_{j-1} \frac{[j-1]_q}{\varphi_{j-1}^{1/s^*} q^{j-1}} \right)$$

$$= \sum_{j=1}^{n} \frac{p^n [j]_p \lambda_j}{[n]_p [n+1]_p} \frac{[j+1]_q}{\varphi_j^{1/s^*} q^j} T_j - \sum_{j=0}^{n-1} \frac{p^n [j+1]_p \lambda_{j+1}}{[n]_p [n+1]_p} \frac{[j]_q}{\varphi_j^{1/s^*} q^j} T_j$$

$$= \varphi_n^{-1/s^*} \frac{p^n [n+1]_q}{q^n [n+1]_p} \lambda_n T_n$$

$$+ \sum_{j=1}^{n-1} \frac{\varphi_j^{-1/s^*} p^n}{q^j [n]_p [n+1]_p} \left(\lambda_j [j]_p [j+1]_q - \lambda_{j+1} [j+1]_p [j]_q \right) T_j,$$

$$t_0 = \varphi_0^{-1/s^*} \lambda_0 T_0.$$

Hence, it can be written that

$$t_n = \sum_{j=0}^{n} a_{nj} T_j$$

where

$$a_{nj} = \begin{cases} \varphi_n^{-1/s^*} \dfrac{p^n [n+1]_q}{q^n [n+1]_p} \lambda_n, & j = n \\ \dfrac{\varphi_j^{-1/s^*} p^n}{q^j [n]_p [n+1]_p} \left(\lambda_j \Delta_{pq}(j) + \Delta \lambda_j [j+1]_p [j]_q \right), & 1 \le j \le n-1 \\ 0, & j > n. \end{cases}$$

So, it follows from Lemma 1 and Lemma 2 that $(t_n) \in l$ whenever $(T_n) \in l_s$ if and only if the matrix $A \in (l_s, l)$ or equivalent the condition

$$\sum_{j=0}^{\infty} \left(\left| \varphi_j^{-1/s^*} \frac{p^j [j+1]_q}{q^j [j+1]_p} \lambda_j \right| + \sum_{n=j+1}^{\infty} \left| \frac{\varphi_j^{-1/s^*} p^n}{q^j [n]_p [n+1]_p} \left(\lambda_j \Delta_{pq}(j) + \Delta \lambda_j [j+1]_p [j]_q \right) \right| \right)^{s^*} < \infty \tag{3}$$

holds.

Here, if we consider the series $\sum_{n=j+1}^{\infty} \frac{p^n}{[n]_p [n+1]_p}$ as follows

$$\sum_{n=j+1}^{\infty} \frac{p^n}{[n]_p [n+1]_p} = \begin{cases} \displaystyle\sum_{n=j+1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right), & p = 1 \\ \displaystyle\sum_{n=j+1}^{\infty} \left(\frac{1}{1-p^n} - \frac{1}{1-p^{n+1}} \right), & p < 1 \text{ or } p > 1, \end{cases}$$

the sum of the series is obtained as follows for each value of p:

$$\sigma_{j+1}^{(p)} = \begin{cases} \dfrac{1}{j+1}, & p = 1 \\ \dfrac{p^{j+1}}{[j+1]_p}, & p < 1 \\ \dfrac{1}{[j+1]_p}, & p > 1. \end{cases}$$

So, the condition (3) is reduced to

$$\sum_{j=0}^{\infty}\left(\left|\varphi_j^{-1/s^*}\frac{p^j[j+1]_q}{q^j[j+1]_p}\lambda_j\right| + \left|\frac{\sigma_{j+1}^{(p)}}{\varphi_j^{1/s^*}q^j}\left(\lambda_j\Delta_{pq}(j) + \Delta\lambda_j[j+1]_p[j]_q\right)\right|\right)^{s^*} < \infty.$$

Theorem 2. Let $1 \le s < \infty$. The necessary and sufficient condition for $\lambda \epsilon(|C^p,\delta|, |C^q,\varphi|_s)$ is

$$\sup_j\left\{\left|\varphi_j^{1/s^*}\frac{q^j[j+1]_p}{p^j[j+1]_q}\lambda_j\right|^s + \sum_{n=j+1}^{\infty}\left|\frac{\varphi_n^{1/s^*}q^n}{p^j[n]_q[n+1]_q}\left(\lambda_j\Delta_{qp}(j) + \Delta\lambda_j[j+1]_q[j]_p\right)\right|^s\right\} < \infty. \quad (4)$$

Proof.

Assume that $\hat{T}_n$ and $\hat{t}_n$ are the transformation sequences of $|C^q,\varphi|_s$ and $|C^p,\delta|$ means of series $\sum\lambda_n u_n$ and $\sum u_n$, respectively. Using the inverse transformation of $\hat{t}_n$, it is obtained that for all $n \ge 1$:

$$\hat{T}_n = \sum_{j=1}^{n}\frac{\varphi_n^{1/s^*}q^n[j]_q}{[n]_q[n+1]_q}\lambda_j u_j$$

$$= \sum_{j=1}^{n}\frac{\varphi_n^{1/s^*}q^n[j]_q}{[n]_q[n+1]_q}\lambda_j\left(\hat{t}_j\frac{[j+1]_p}{p^j} - \hat{t}_{j-1}\frac{[j-1]_p}{p^{j-1}}\right)$$

$$= \varphi_n^{1/s^*}\sum_{j=1}^{n}\frac{q^n[j]_q}{[n]_q[n+1]_q}\lambda_j\,\hat{t}_j\frac{[j+1]_p}{p^j} - \varphi_n^{1/s^*}\sum_{j=0}^{n-1}\frac{q^n[j+1]_q}{[n]_q[n+1]_q}\hat{t}_j\lambda_{j+1}\frac{[j]_p}{p^j}$$

$$= \varphi_n^{1/s^*}\frac{q^n[n+1]_p}{p^n[n+1]_q}\lambda_n\hat{t}_n + \varphi_n^{1/s^*}\sum_{j=1}^{n-1}\frac{q^n}{p^j[n]_q[n+1]_q}\left(\lambda_j[j]_q[j+1]_p - \lambda_{j+1}[j+1]_q[j]_p\right)\hat{t}_j,$$

$$\hat{T}_0 = \varphi_0^{1/s^*} \lambda_0 \hat{t}_0.$$

So, it can be written that

$$\hat{T}_n = \sum_{j=0}^{n} b_{nj}\, \hat{t}_j,$$

where

$$b_{nj} = \begin{cases} \varphi_n^{1/s^*} \dfrac{q^n [n+1]_p}{p^n [n+1]_q} \lambda_n, & j = n \\ \varphi_n^{1/s^*} \dfrac{q^n}{p^j [n]_q [n+1]_q} \left(\lambda_j \Delta_{qp}(j) + \Delta\lambda_j [j+1]_q [j]_p \right), & 1 \le j \le n-1 \\ 0, & j > n. \end{cases}$$

It follows from Lemma 3 that the condition (4) is immediately obtained which completes the proof.

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