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Researches and Evaluations in the Field of *MATHEMATICS*

March
2025

İmtiyaz Sahibi • Yaşar Hız
Genel Yayın Yönetmeni • Eda Altunel
Yayına Hazırlayan • Gece Kitaplığı
Editör • Doç. Dr. Nil MANSUROĞLU

Birinci Basım • Mart 2025 / ANKARA

ISBN • 978-625-388-268-6

Gece Kitaplığı
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Sertifika No: 42488

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İÇİNDEKİLER

CHAPTER 1

A STUDY ON NOISY SIGNALS ENHANCEMENT FOR TWO SEPERATE ITERATIVE ALGORITHMS

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Introduction

Let $\mathcal{M} \neq \emptyset$ be a subset of a Banach space $\mathfrak{I}$. Let $\Omega: \mathcal{M} \to \mathcal{M}$ be a map. A point $z^* \in \mathcal{M}$ is a fixed point of Ω iff $\Omega z^* = z^*$. Let

$$F_{\Omega} := \{z^* \in \mathcal{M}: \Omega z^* = z^*\}.$$

Ω called to be nonexpansive if

$$\|\Omega \mathrm{x} - \Omega \mathrm{y}\| \leq \|x - y\|$$

for each $x, y \in \mathcal{M}$.

Fixed point theory is a field of mathematics that serves as a highly influential tool, with widespread applications across various disciplines, including economics (McLennan, 2018), game theory (Barron, 2024), computer science (Alghamdi, et al., 2013), physics (Tycko, et al., 1985), biology (Marrow, et al., 1996), and engineering (Bauschke, et al., 2011).

Several iterative methods are commonly employed to estimate the fixed points of some maps. Prominent instances of such methods involve the Picard (Picard, 1890), Mann (Mann, 1953), Ishikawa (Ishikawa, 1974), and S- iterations (Agarwal, et al., 2007). A significant number of researchers have concentrated on improving these methods in the last few years. Lately, a distinct novel fixed point iterative technique was proposed by Alam&Rohen (Alam and Rohen 2025) and Filali et al. (Filali, et al., 2024) to estimate the fixed points of contraction maps in a Banach space.

Assume that $\mathcal{M} \neq \emptyset$ is a set and $\Omega: \mathcal{M} \to \mathcal{M}$ be a map. Then following are defined as:

Definition 1. The iteration method of Alam&Rohen is given as:

$$\begin{aligned} x_{n+1} &= \Omega^2 r_{3n}, \\ r_{3n} &= \Omega(t_{3n}\Omega r_{2n} + (1 - t_{3n})r_{2n}), \\ r_{2n} &= \Omega(t_{2n}\Omega r_{1n} + (1 - t_{2n})r_{1n}), \\ r_{1n} &= \Omega(t_{1n}\Omega x_n + (1 - t_{1n})x_n), \end{aligned} \tag{1}$$

for sequences $\{t_{1n}\}$, $\{t_{1n}\}$, $\{t_{1n}\} \subset (0,1)$ (Alam and Rohen 2025).

Definition 2. The iteration method of Filali et al. is defined as:

$$
\begin{aligned}
x_{n+1} &= \Omega\big((1-t_{3n})\Omega r_{2n} + t_{3n}\Omega r_{3n}\big) \qquad (2)\\
r_{3n} &= \Omega\big((1-t_{2n})\Omega r_{2n} + t_{2n}\Omega r_{1n}\big),\\
r_{2n} &= \Omega^2 r_{1n},\\
r_{1n} &= \Omega((1-t_{1n})x_n + t_{1n}\Omega x_n),
\end{aligned}
$$

for sequences $\{t_{1n}\}$, $\{t_{1n}\}$, $\{t_{1n}\} \subset (0,1)$ (Filali, et al., 2024).

Subsequently, we will outline fundamental concepts that will be utilized in the second part.

Definition 3. Let $\mathfrak{J}$ be a Banach space with dimension $\mathfrak{J} \geq 2$. The modulus of $\mathfrak{J}$ is the map $\delta_{\mathfrak{J}}: (0,2] \to [0,1]$, which is characterized by

$$\delta_{\mathfrak{J}}(\varepsilon) = inf\left\{1 - \left\|\frac{y+x}{2}\right\| : \|x\| = 1 = \|y\|, \varepsilon = \|-x+y\| \right\}$$

(Aksoy and Khamsi 1990).

Lemma 4. Let $\{\omega_n\}$ and $\{\varsigma_n\}$ denote two sequences of real numbers that are both nonnegative, which fulfill the inequality

$$\omega_{n+1} \leq \omega_n + \varsigma_n$$

for every $n \geq 1$. If $\sum_{n=1}^{\infty} \varsigma_n < \infty$, then $\lim_{n\to\infty} \omega_n$ exists (Tan and Xu, 1993).

Based on the information above, the structure of this paper has been outlined as follows: In Sect. 2, we offer some convergence theorems for a nonexpansive map in a Banach space. In Sect. 3, The algorithms presented by Alam&Rohen and Filali et al. are utilized to handle the issues of the signal refinement (or denoising), with a comparative approach.

Fixed Point Results

Lemma 5. Let $\mathfrak{J}$ be a real Banach space, $\mathcal{M} \neq \emptyset$ be a closed convex subset of a Banach space $\mathfrak{J}$, $\Omega: \mathcal{M} \to \mathcal{M}$ be a nonexpansive map, and $F_\Omega \neq \emptyset$. Let $\{t_{1n}\}$, $\{t_{2n}\}$, $\{t_{3n}\} \subset (0,1)$. Let $\{x_n\}$ be identified by (1). Then $\lim_{n\to\infty} \|x_n - z^*\|$ exists for every $z^* \in F_\Omega$.

Proof. Let $z^* \in F_\Omega$. It follows from (1) and nonexpansiveness of Ω that

$$\begin{aligned}\|x_{n+1}-z^*\| &\le \|\Omega^2 r_{3n}-z^*\| \\ &\le \|\Omega r_{3n}-z^*\| \\ &\le \|r_{3n}-z^*\| \\ &= \|\Omega(t_{3n}\Omega r_{2n}+(1-t_{3n})r_{2n})-z^*\| \\ &\le \|t_{3n}\Omega r_{2n}+(1-t_{3n})r_{2n}-z^*\| \\ &\le t_{3n}\|\Omega r_{2n}-z^*\|+(1-t_{3n})\|r_{2n}-z^*\| \\ &\le \|r_{2n}-z^*\| \\ &= \|\Omega(t_{2n}\Omega r_{1n}+(1-t_{2n})r_{1n})-z^*\| \\ &\le \|(t_{2n}\Omega r_{1n}+(1-t_{2n})r_{1n})-z^*\| \\ &\le t_{2n}\|\Omega r_{1n}-z^*\|+(1-t_{2n})\|r_{1n}-z^*\| \\ &\le \|r_{1n}-z^*\| \\ &= \|\Omega(t_{1n}\Omega x_n+(1-t_{1n})x_n)-z^*\| \\ &\le \|t_{1n}\Omega x_n+(1-t_{1n})x_n-z^*\| \\ &\le t_{1n}\|\Omega x_n-z^*\|+(1-t_{1n})\|x_n-z^*\| \\ &\le \|x_n-z^*\|.\end{aligned} \tag{3}$$

Owing to Lemma 4, we obtain $\lim_{n\to\infty}\|x_n-z^*\|$ exists. Notably, $\{x_n\}$ is bounded.

Theorem 6. Let $\mathfrak{J}$ be a uniformly convex Banach space, $\mathcal{M},\Omega$ and $\{x_n\}$ be held as in Lemma 5. Then $\{x_n\}$ converges to a point of F_Ω iff $liminf_{n\to\infty}d(x_n,F_\Omega)=0$, where $d(x_n,F_\Omega)$ holds the distance of x to set F_Ω, hence $d(x_n,F_\Omega)=inf_{y\in F_\Omega}d(x,y)$.

Proof. Necessity. Given that (3) holds for each $z^*\in F_\Omega$, we reach

$$d(x_n,F_\Omega)\le d(x_{n-1},F_\Omega),$$

for all $n\ge n_0$, Lemma 5 that $\lim_{n\to\infty}d(x_n,F_\Omega)$ exists, thus $\lim_{n\to\infty}d(x_n,F_\Omega)=0$.

Sufficiency. Next, we prove that $\{x_n\}\subset\mathcal{M}$ is a Cauchy sequence. Because $\lim_{n\to\infty}d(x_n,F_\Omega)=0$, there exists n_0 in N such that for each $n\ge n_0$, $d(x_n,F_\Omega)<\frac{\varepsilon}{2}$ for given $\varepsilon>0$. In particular,

$$inf\{\|x_{n_0}-z^*\|: z^*\in F_\Omega\}<\frac{\varepsilon}{2}.$$

Therefore, there is $z^*\in F_\Omega$ such that $\|x_{n_0}-z^*\|<\frac{\varepsilon}{2}$. Now, for $m,n\ge n_0$,

$$\begin{aligned}\|x_n-x_{m+n}\| &\le \|x_{m+n}-z^*\|+\|x_n-z^*\| \\ &\le \|x_{n_0}-z^*\|\end{aligned}$$

$$< \varepsilon.$$

Hence $\{x_n\} \subset \mathcal{M}$ is a Cauchy sequence. As $\mathcal{M}$ is closed in the Banach space $\mathfrak{I}$, there is a point z^* in $\mathcal{M}$ such that $\lim_{n\to\infty} x_n = z^*$. Now,

$$\lim_{n\to\infty} d(x_n, F_\Omega) = 0 \text{ gives that } d(x_n, F_\Omega) = 0.$$

Due to closed of F, we attain $z^* \in F_\Omega$.

Lemma 7. Let $\mathfrak{I}$ be a real Banach space, $\mathcal{M} \neq \emptyset$ be a closed convex subset of a Banach space $\mathfrak{I}$, $\Omega: \mathcal{M} \to \mathcal{M}$ be a nonexpansive map, and $F_\Omega \neq \emptyset$. Let $\{t_{1n}\}$, $\{t_{2n}\}$, $\{t_{3n}\} \subset (0,1)$. Let $\{x_n\}$ be identified by (2). Then $\lim_{n\to\infty} \|x_n - z^*\|$ exists for every $z^* \in F_\Omega$.

Proof. Let $z^* \in F_\Omega$. By (2), we get

$$\begin{aligned} \|r_{1n} - z^*\| &= \|\Omega((1-t_{1n})x_n + t_{1n}\Omega x_n) - z^*\| \\ &\leq \|(1-t_{1n})x_n + t_{1n}\Omega x_n - z^*\| \\ &\leq (1-t_{1n})\|x_n - z^*\| + t_{1n}\|\Omega x_n - z^*\| \\ &\leq \|x_n - z^*\| \end{aligned} \tag{4}$$

from (4), we have,

$$\begin{aligned} \|r_{2n} - z^*\| &= \|\Omega^2 r_{1n} - z^*\| \\ &\leq \|\Omega r_{1n} - z^*\| \\ &\leq \|r_{1n} - z^*\|. \end{aligned} \tag{5}$$

and by (4) and (5), we attain,

$$\begin{aligned} \|r_{3n} - z^*\| &= \left\|\Omega\big((1-t_{2n})\Omega r_{2n} + t_{2n}\Omega r_{1n}\big) - z^*\right\| \\ &\leq \|(1-t_{2n})\Omega r_{2n} + t_{2n}\Omega r_{1n} - z^*\| \\ &\leq (1-t_{2n})\|\Omega r_{2n} - z^*\| + t_{2n}\|\Omega r_{1n} - z^*\| \\ &\leq (1-t_{2n})\|r_{2n} - z^*\| + t_{2n}\|r_{1n} - z^*\| \\ &\leq \|r_{1n} - z^*\|. \end{aligned} \tag{6}$$

In the end, by virtue of (5) and (6), we point out that

$$\begin{aligned} \|x_{n+1} - z^*\| &= \left\|\Omega\big((1-t_{3n})\Omega r_{2n} + t_{3n}\Omega r_{3n}\big) - z^*\right\| \\ &\leq \|(1-t_{3n})\Omega r_{2n} + t_{3n}\Omega r_{3n} - z^*\| \\ &\leq (1-t_{3n})\|\Omega r_{2n} - z^*\| + t_{3n}\|\Omega r_{3n} - z^*\| \\ &\leq (1-t_{3n})\|r_{2n} - z^*\| + t_{3n}\|r_{3n} - z^*\| \\ &\leq \|r_{1n} - z^*\|. \end{aligned} \tag{7}$$

which exhibit that $\{\|x_{n+1} - z^*\|\}$ is decreasing and bounded. Therefore, $\lim_{n\to\infty} \|x_n - z^*\|$ exists.

Theorem 8. Let $\Im$ be a uniformly convex Banach space, $\mathcal{M}, \Omega$ and $\{x_n\}$ be held as in Lemma 7. Then $\{x_n\}$ converges to a point of F_Ω iff $liminf_{n\to\infty} d(x_n, F_\Omega) = 0$, where $d(x_n, F_\Omega)$ holds the distance of x to set F_Ω, hence $d(x_n, F_\Omega) = inf_{y\in F_\Omega} d(x, y)$.

Proof. Necessity. Since (7) holds for each $z^* \in F_\Omega$, we reach

$$d(x_n, F_\Omega) \leq d(x_{n-1}, F_\Omega),$$

for all $n \geq n_0$, Lemma 5 that $\lim_{n\to\infty} d(x_n, F_\Omega)$ exists, thus $\lim_{n\to\infty} d(x_n, F_\Omega) = 0$.

Sufficiency. We now show that $\{x_n\} \subset \mathcal{M}$ is a Cauchy sequence. As $\lim_{n\to\infty} d(x_n, F_\Omega) = 0$, there exists n_0 in N such that for each $n \geq n_0$, $d(x_n, F_\Omega) < \frac{\varepsilon}{2}$ for given $\varepsilon > 0$. In particular,

$$inf\{\|x_{n_0} - z^*\| : z^* \in F_\Omega\} < \frac{\varepsilon}{2}.$$

Therefore, there is $z^* \in F_\Omega$ such that $\|x_{n_0} - z^*\| < \frac{\varepsilon}{2}$. Now, for $m, n \geq n_0$,

$$\begin{aligned} \|x_n - x_{m+n}\| &\leq \|x_{m+n} - z^*\| + \|x_n - z^*\| \\ &\leq \|x_{n_0} - z^*\| \\ &< \varepsilon. \end{aligned}$$

Thus $\{x_n\} \subset \mathcal{M}$ is a Cauchy sequence. As $\mathcal{M}$ is closed in the Banach space $\Im$, there is a point z^* in $\mathcal{M}$ such that $\lim_{n\to\infty} x_n = z^*$. Now,

$$\lim_{n\to\infty} d(x_n, F_\Omega) = 0 \text{ gives that } d(x_n, F_\Omega) = 0.$$

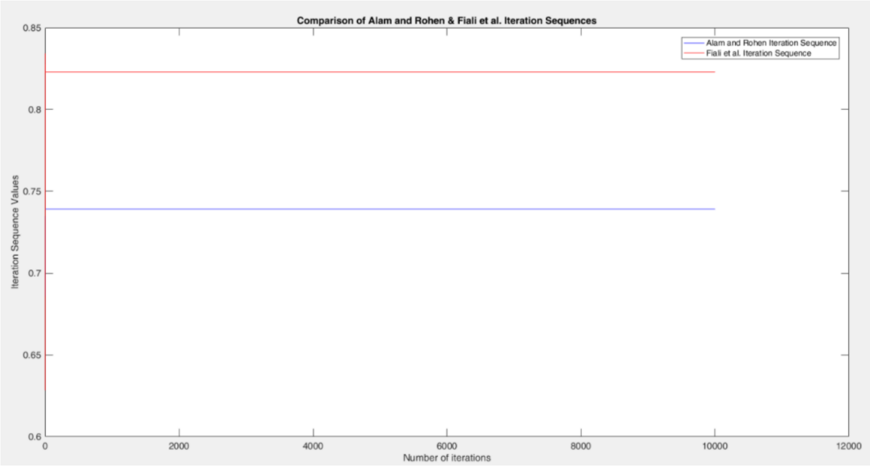
Due to closed of F, we attain $z^* \in F_\Omega$.

Example 9. Let $\Im = R$ and $\mathcal{M} = [-1,1]$. Define a map $\Omega: \mathcal{M} \to \mathcal{M}$ by $\Omega x = \cos x$ for $x \in \mathcal{M}$. It is clear to prove that Ω is a nonexpansive map, $F_\Omega = \{0.739085\}$. Let $t_{1n} = t_{2n} = t_{3n} = \frac{23}{48}$ and $x_1 = \frac{\pi}{5}$. Numerical computations were done with Matlab R2016. By Figure 1 and Table 1, it is evident that (1) converge to $0.739085 \in F_\Omega$ and (2) converge to $0.8229 \notin F_\Omega$. This indicates that the Theorem 6 is applicable, while Theorem 8 is not.

Table 1: A numerical analysis comparing the iteration methods of Filali et al. and Alam & Rohen for the Example

n	(1) iteration	(2) iteration
1	0,6283	0,6283
2	0.7392	0.8343
3	0.7391	0.8221
4	0.7391	0.8230
5	0.7391	0.8229
6	0.7391	0.8229
7	0.7391	0.8229
8	0.7391	0.8229
9	0.7391	0.8229
10	0.7391	0.8229
⋮	⋮	⋮
1000	0.7391	0.8229
⋮	⋮	⋮
2000	0.7391	0.8229
⋮	⋮	⋮
3000	0.7391	0.8229
⋮	⋮	⋮
4000	0.7391	0.8229
⋮	⋮	⋮
5000	0.7391	0.8229
⋮	⋮	⋮
10000	0.7391	0.8229
⋮	⋮	⋮

Figure 1: A numerical analysis comparing the iteration methods of Filali et al. and Alam & Rohen for the Example 9



The signal enhancement

Signal enhancement, which aims to improve the quality of a signal by minimizing interference, increasing clarity, or emphasizing specific details, is commonly applied when the signal is compromised by noise, degradation, or weakness, and it is often achieved through iterative algorithms that refine a solution through a series of repetitive steps.

Motivated this facts, we created a code using the (1) and (2) iterative technique to obtain a cleaner (smoothed) version of a noisy signal. Here, the aim is to improve the signal by using a low pass filter (moving average) in each step in every two iterations.

$t_{1n} = t_{2n} = t_{3n} = beta = 0.5$ in (1) and (2).

Algorithm (Filali et al.)

Input: Time vector (t), original signal (original_signal), noise (noise), noisy signal (noisy_signal), and filtering parameters (alpha, num_iterations).

Output: Original signal, noisy signal, and the improved signal after filtering (x_k). These outputs are visualized in plots.

1 for k = 1:num_iterations

2 $x_{n+1} = \Omega\big((1 - beta)\Omega r_{2n} + beta\Omega r_{3n}\big)$

3 $r_{3n} = \Omega\big((1 - beta)\Omega r_{2n} + beta\Omega r_{1n}\big)$

4 $r_{2n} = \Omega^2 r_{1n}$

5 $r_{1n} = \Omega((1 - beta)x_n + beta\Omega x_n)$

6 end.

Algorithm (Alam & Rohen)

Input: Time vector (t), original signal (original_signal), noise (noise), noisy signal (noisy_signal), and filtering parameters (alpha, num_iterations).

Output: Original signal, noisy signal, and the improved signal after filtering (x_k). These outputs are visualized in plots.

1 for

2 $x_{n+1} = \Omega^2 r_{3n}$

3 $r_{3n} = \Omega(beta\Omega r_{2n} + (1 - beta)r_{2n})$

4 $r_{2n} = \Omega(beta\Omega r_{1n} + (1 - beta)r_{1n})$

5 $r_{1n} = \Omega(beta\Omega x_n + (1 - beta)x_n)$

6 end.

These specific implementations are modified versions of the Filali et al. and Alam & Rohen iteration method, utilizing a moving average filter and multiple iterations to minimize noise (see, Fig 2 and Fig 3).

Figure 2: Visualize results for Alam & Rohen iteration

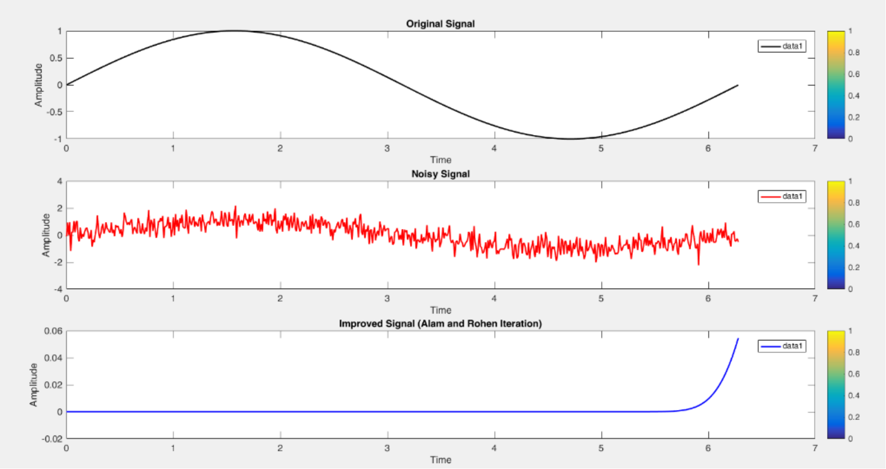
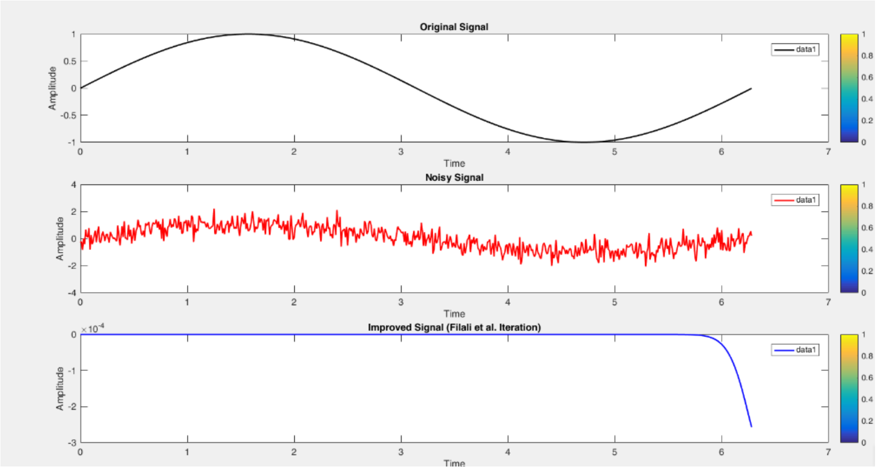


Figure 3: Visualize results for Filali et al. iteration



When Figure 2 and Figure 3 are examined, it is seen that the amplitude of the Alam & Rohen iteration takes values greater than 0 over time, while the amplitude of the Filali et al. iteration takes values less than 0 over time.

In signal processing, the amplitude being less than zero and greater than zero over time refers to the state of the amplitude being positive and negative. These two states are related to different parts of the signal.

Amplitude less than 0, i.e. Negative Amplitude, means that the signal is changing in a negative direction. In this case, the phase, transformation or polarity of the signal is reversed. Amplitude less than zero indicates that the signal is biased downwards. This means that the lower half of the waveform is negative, especially in analog signals.

If the amplitude is greater than 0, that is, positive amplitude, it means that the signal changes in the positive direction. In this case, the phase or polarity of the signal is upward. If the amplitude is greater than zero, it indicates that the signal tends upward. In other words, it indicates that the upper half of the signal is positive.

Consequently, changes in these amplitudes over time may indicate modulation of the signal or the influence of different frequency components. Such changes can be especially important in audio, video and communication systems.

Conclusion

In this paper, based on the aforementioned concepts, convergence theorems have been examined for two new iteration methods introduced by Filali et al. and Alam & Rohen. Additionally, the signal enhancement challenges are analyzed for these iteration approaches.

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CHAPTER 2

ASSOCIATED CURVES OF SPECIAL SMARANDACHE CURVES IN THE CONTEXT OF THE TYPE-2 BISHOP FRAME

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ASSOCIATED CURVES OF SPECIAL SMARANDACHE CURVES IN THE CONTEXT OF THE TYPE-2 BISHOP FRAME

Esra DAMAR[1]

1. INTRODUCTION

One of the fundamental areas of differential geometry involves characterizing curves in various spaces, such as Euclidean and Minkowski spaces, using different frame systems, including the Frenet and Bishop frames. While the Frenet-Serret frame is widely used in curve analysis, it becomes undefined at points where the second derivative of the curve is zero. This limitation has led to the development of alternative orthonormal frames, such as the Bishop frame, which provides a more flexible approach, particularly when torsion is zero or minimal [1]. Helices, commonly appearing in various applications, are typically classified as general helices, cylindrical helices, or slant helices. A key property of helices is that their tangent vector field forms a constant angle with a fixed direction [2]. Lancret's theorem states that the ratio of torsion to curvature is constant for helices, a concept that has been extended to general helices through studies involving Killing vector fields [3-5]. Slant helices, characterized by a constant angle between the principal normal vector and a fixed direction, have also been studied extensively in different frames [6].

Another significant class of curves in differential geometry is integral curves, which arise as solutions to differential equations and are often used to define new parameterized curves. Special curves such as Bertrand, Mannheim, and Smarandache curves have been explored in various frames due to their applications in physics, engineering, and computer graphics [7-8].

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Smarandache curves, in particular, provide a deeper geometric interpretation by transitioning between different frames. Their applications span general relativity, robotics, and structural analysis, making them an essential tool in both theoretical and applied research [9-15]. In this paper, new adjoint curves are introduced by combining special Smarandache curves with integral curves in the type-2 Bishop frame. These adjoint curves are systematically defined, and their geometric properties are examined. Relationships between the original curve and its adjoint curves are established, leading to necessary and sufficient conditions for a curve to be a general or slant helix.

2. PRELİMİNARİES

This section introduces essential concepts required for the following sections. Let $\alpha = \alpha(s)$ be differentiable unit speed curves in E^3 and its Frenet apparatus be $\{\mathbf{T}, \mathbf{N}, \mathbf{B}, \kappa, \tau\}$. When α is a unit speed curve, its unit tangent vector is $\mathbf{T}(s) = \alpha'(s)$ and its curvature is $\kappa = \|\alpha''(s)\|$. $\alpha''(s) = \kappa(s)\mathbf{N}(s)$ and $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$ gives the principal normal vector and the unit binormal vector of α respectively. Next, the well-known Frenet formula is shown as

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} \tag{1}$$

where $\tau(s) = \langle \mathbf{N}'(s), \mathbf{B}(s) \rangle$ is the torsion of α. The derivative formula of the type-2 Bishop frame is given as follows:

$$\begin{pmatrix} \mathbf{M}_1' \\ \mathbf{M}_2' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\epsilon_1 \\ 0 & 0 & -\epsilon_2 \\ \epsilon_1 & \epsilon_2 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{M}_1 \\ \mathbf{M}_2 \\ \mathbf{B} \end{pmatrix} \tag{2}$$

The set $\{\mathbf{M}_1, \mathbf{M}_2, \mathbf{B},\}$ is referred to as type-2 Bishop trihedron in this case, the curvatures ϵ_1 and ϵ_2 are called Bishop curvatures. The relation matrix can be expressed

$$\begin{aligned} \mathbf{T} &= \sin\theta(s)\mathbf{M}_1 - \cos\theta(s)\mathbf{M}_2 \\ \mathbf{N} &= \cos\theta(s)\mathbf{M}_1 + \sin\theta(s)\mathbf{M}_2 \\ \mathbf{B} &= \mathbf{B} \end{aligned} \tag{3}$$

where $\theta(s) = \arctan\left(\frac{\epsilon_2}{\epsilon_1}\right)$, and $\tau = \sqrt{\epsilon_1^2 + \epsilon_2^2}$, $\theta = \int_0^s \kappa(s)ds$ [16]. Here, type-2 Bishop curvatures are defined by

$$\epsilon_1(s) = -\tau\cos\theta(s), \quad \epsilon_2(s) = -\tau\sin\theta(s). \tag{4}$$

On the other hand, if Eq (3) are regularized, the following equations are obtained

$$\begin{aligned} \mathbf{M_1} &= \sin\theta(s)\mathbf{T} + \cos\theta(s)\mathbf{N}, \\ \mathbf{M_2} &= -\cos\theta(s)\mathbf{T} + \sin\theta(s)\mathbf{N}, \\ \mathbf{B} &= \mathbf{B}. \end{aligned} \tag{5}$$

When the angle between a curve's tangent lines and a fixed direction remains constant, the curve is called to general helix. The general helix's axis is the name given to this fixed direction. In 1802, Lancret articulated the definition of a helix, stating that a curve may be classified only as a general helix if the harmonic curvature or the ratio $\frac{\tau}{\kappa}$ remains constant, with $\kappa \neq 0$, if both $\kappa \neq 0$, and $\tau \neq 0$ are constants, the general helix is referred to as a circular helix [3].

The constant geodesic curvature function of the principal image of the constant normal indicatrix characterizes a slant helix, as stated in [6]. This function that is constant is provided by

$$\sigma(s) = \left(\frac{\kappa^2}{(\kappa^2 + \tau^2)^{\frac{3}{2}}} \left(\frac{\tau}{\kappa}\right)'(s) \right).$$

Theorem 2.1.1 ([17]). Let the unit speed curve $\alpha: I \to E^3$ be a slant helix with non-zero natural type-2 Bishop curvatures. Then α is slant helix if and only if $\frac{\epsilon_2}{\epsilon_1}$ is constant.

Definition 2.1.2 ([18]). Let α be an *s*-arc length parameterized regular curve with nonvanishing torsion and $\{\mathbf{T}_\alpha, \mathbf{N}_\alpha, \mathbf{B}_\alpha\}$ is the Frenet frame of α the adjoint curve of α is defined as

$$\beta(s) = \int_{s_0}^{s} \mathbf{B}_\alpha(u)du.$$

Definition 2.1.3 ([15]). Let α be an *s*-arc length parameterized regular curve with nonvanishing torsion and $\{\mathbf{T}_\alpha, \mathbf{N}_\alpha, \mathbf{B}_\alpha\}$ is the Frenet frame of α. Smarandache TN, NB, and TNB curves are defined by

$$\beta = \frac{1}{\sqrt{2}}(\mathbf{T}_\alpha + \mathbf{N}_\alpha),$$

$$\gamma = \frac{1}{\sqrt{2}}(\mathbf{N}_\alpha + \mathbf{B}_\alpha),$$

$$\mu = \frac{1}{\sqrt{3}}(\mathbf{T}_\alpha + \mathbf{N}_\alpha + \mathbf{B}_\alpha)$$

respectively.

3. MAIN RESULTS

3.1. Type-2 Bishop Adjoint Curves

Definition 3.1.1. Let α be an *s*-arc length parameterized regular curve with type-2 Bishop apparatus $\{\mathbf{M}_1^\alpha, \mathbf{M}_2^\alpha, \mathbf{B}_\alpha, \epsilon_1^\alpha, \epsilon_2^\alpha\}$. The following curves, derived from the type-2 Bishop frame of α, are defined as its type-2 Bishop adjoint curves:

The $\mathbf{M}_1^\alpha\mathbf{B}_\alpha$-type-2 Bishop adjoint curve:

$$\beta(\tilde{s}) = \frac{1}{\sqrt{2}}\int\big(\mathbf{M}_1^\alpha(s) + \mathbf{B}_\alpha(s)\big)ds \tag{6}$$

The $\mathbf{M}_2^\alpha\mathbf{B}_\alpha$- type-2 Bishop adjoint curve:

$$\gamma(\tilde{s}) = \frac{1}{\sqrt{2}}\int\big(\mathbf{M}_2^\alpha(s) + \mathbf{B}_\alpha(s)\big)ds \tag{7}$$

The $\mathbf{M}_1^\alpha\mathbf{M}_2^\alpha$ -type-2 Bishop adjoint curve:

$$\Omega(\tilde{s}) = \frac{1}{\sqrt{2}}\int\big(\mathbf{M}_1^\alpha(s) + \mathbf{M}_2^\alpha(s)\big)ds. \tag{8}$$

Each of these adjoint curves is obtained through specific linear combinations of the frame vectors, capturing different geometric relationships within the type-2 Bishop frame.

Remark 3.1.2. ([1]). Let α be a regular curve parameterized by arc length *s,* and let β, γ, Ω be its adjoint curves. The arc length parameter of $\tilde{s}$ these adjoint curves can be chosen such that $s = \tilde{s}$.

3.2. $\mathbf{M}_1^{\alpha}\mathbf{B}_{\alpha}$ -type-2 Bishop adjoint curve

Theorem 3.2.1. Let α be an *s*-arc length parameterized regular curve in E^3 with Bishop apparatus $\{\mathbf{M}_1^{\alpha}, \mathbf{M}_2^{\alpha}, \mathbf{B}_{\alpha}, \epsilon_1^{\alpha}, \epsilon_2^{\alpha}\}$. The Frenet vector fields, curvature and torsion of β are given by

$$\mathbf{T}_{\beta} = \frac{1}{\sqrt{2}}(\mathbf{M}_1^{\alpha} + \mathbf{B}_{\alpha}), \tag{9}$$

$$\mathbf{N}_{\beta} = \frac{1}{\sqrt{1+2g^2}}(g\mathbf{M}_1^{\alpha} + \mathbf{M}_2^{\alpha} - g\mathbf{B}_{\alpha}), \tag{10}$$

$$\mathbf{B}_{\beta} = \frac{1}{\sqrt{1+4g^2}}(-\mathbf{M}_1^{\alpha} + 2g\mathbf{M}_2^{\alpha} + \mathbf{B}_{\alpha}), \tag{11}$$

$$\kappa_{\beta} = \frac{1}{\sqrt{2}}\epsilon_2^{\alpha}\sqrt{1+2g^2} \tag{12}$$

$$\tau_{\beta} = -\frac{1}{\sqrt{2}}\epsilon_2^{\alpha} - \frac{\sqrt{2}g'}{1+2g^2}, \tag{13}$$

where $g = \frac{\epsilon_1^{\alpha}}{\epsilon_2^{\alpha}}$.

Proof 3.2.2 Differentiating Eq (7) and applying the Frenet formulas, we obtain

$$\frac{d\beta}{d\tilde{s}}\frac{d\tilde{s}}{ds} = \frac{1}{\sqrt{2}}(\mathbf{M}_1^{\alpha} + \mathbf{B}_{\alpha}),$$

$$\mathbf{T}_{\beta}\frac{d\tilde{s}}{ds} = \frac{1}{\sqrt{2}}(\mathbf{M}_1^{\alpha} + \mathbf{B}_{\alpha}),$$

taking the norm on both sides gives:

$$\frac{d\tilde{s}}{ds} = 1,$$

thus, we conclude:

$$\mathbf{T}_{\beta} = \frac{1}{\sqrt{2}}(\mathbf{M}_1^{\alpha} + \mathbf{B}_{\alpha}). \tag{14}$$

differentiating (14) with respect to s and using Eq (2) we find

$$\mathbf{T}_{\beta}' = \frac{1}{\sqrt{2}}(\epsilon_1^{\alpha}\mathbf{M}_1^{\alpha} + \epsilon_2^{\alpha}\mathbf{M}_2^{\alpha} - \epsilon_1^{\alpha}\mathbf{B}_{\alpha}).$$

From this, the curvature κ_β and principal normal vector $\mathbf{N}_\beta$ of curve β are given by:

$$\kappa_\beta = \left\|\mathbf{T}'_\beta\right\| = \frac{1}{\sqrt{2}}\sqrt{2\epsilon_1^{\alpha^2} + \epsilon_2^{\alpha^2}},$$

$$\mathbf{N}_\beta = \frac{1}{\sqrt{4\epsilon_1^{\alpha^2} + 2\epsilon_2^{\alpha^2}}}(\epsilon_1^\alpha \mathbf{M}_1^\alpha + \boldsymbol{\epsilon}_2^\alpha \mathbf{M}_2^\alpha - \boldsymbol{\epsilon}_1^\alpha \mathbf{B}_\alpha).$$

Next, the binormal vector is determined as:

$$\mathbf{B}_\beta = \mathbf{T}_\beta \times \mathbf{N}_{\beta.}$$

Substituting their values:

$$\mathbf{B}_\beta = \frac{1}{\sqrt{4\epsilon_1^{\alpha^2} + \epsilon_2^{\alpha^2}}}(-\epsilon_2^\alpha \mathbf{M}_1^\alpha + 2\epsilon_1^\alpha \mathbf{M}_2^\alpha - \epsilon_2^\alpha \mathbf{B}_\alpha).$$

To find the torsion τ_β, we differentiate $\mathbf{N}_\beta$ and use:

$$\tau_\beta = \langle \mathbf{N}'_\beta, \mathbf{B}_\beta \rangle.$$

After simplification, we obtain:

$$\tau_\beta = -\frac{1}{\sqrt{2}}\epsilon_2^\alpha - \sqrt{2}\frac{\left(\frac{\epsilon_1^\alpha}{\epsilon_2^\alpha}\right)'}{1 + 2\left(\frac{\epsilon_1^\alpha}{\epsilon_2^\alpha}\right)^2}$$

Setting:

$$g = \frac{\epsilon_1^\alpha}{\epsilon_2^\alpha}$$

we arrive at the final result.

Theorem 3.2.3. Let α be an *s*-arc length parameterized regular curve in E^3,equipped with the type-2 Bishop apparatus $\{M_1^\alpha, M_2^\alpha, B_\alpha, \epsilon_1^\alpha, \epsilon_2^\alpha\}$. Let β be the $\mathbf{M}_1^\alpha\mathbf{B}_\alpha$-type-2 Bishop adjoint curve of α. The type-2 Bishop vector fields and curvatures associated with β are given by:

$$N_1^\beta = \left(\frac{1}{\sqrt{2}}\sin\theta_\beta + \frac{g}{\sqrt{1+2g^2}}\cos\theta_\beta\right)\mathbf{M}_1^\alpha + \frac{\cos\theta_\beta}{\sqrt{1+2g^2}}\mathbf{M}_2^\alpha$$
$$+\left(\tfrac{1}{\sqrt{2}}\sin\theta_\beta - \tfrac{g}{\sqrt{1+2g^2}}\cos\theta_\beta\right)\mathbf{B}_\alpha$$
$$N_2^\beta = \left(-\frac{1}{\sqrt{2}}\cos\theta_\beta + \frac{g}{\sqrt{1+2g^2}}\sin\theta_\beta\right)\mathbf{M}_1^\alpha + \frac{\sin\theta_\beta}{\sqrt{1+2g^2}}\mathbf{M}_2^\alpha$$
$$+\left(\tfrac{1}{\sqrt{2}}\sin\theta_\beta - \tfrac{g}{\sqrt{1+2g^2}}\cos\theta_\beta\right)\mathbf{B}_\alpha$$
$$\epsilon_1^\beta = \left(\frac{1}{\sqrt{2}}\epsilon_2^\alpha + \sqrt{2}\frac{g'}{1+2g^2}\right)\cos\theta_\beta$$
$$\epsilon_2^\beta = \left(\tfrac{1}{\sqrt{2}}\epsilon_2^\alpha + \sqrt{2}\tfrac{g'}{1+2g^2}\right)\sin\theta_\beta.$$

Proof 3.2.3. By substituting Eqs. (9) and (10) into Eqs (4) and (5), the proof follows directly.

Corollary 3.2.4. If the unit speed curve $\alpha: I \to E^3$ be a slant helix with respect to the Bishop frame, then its $\mathbf{M}_1^\alpha\mathbf{B}_\alpha$-type-2 Bishop adjoint curve is a general helix.

Proof 3.2.4. By computing the the ratio of the torsion and curvature of the $\mathbf{M}_1^\alpha\mathbf{B}_\alpha$ -Bishop adjoint curve of α , as given in Theorem 3.2.1., we obtain:

$$\frac{\tau_\beta}{\kappa_\beta} = \frac{-\frac{1}{\sqrt{2}}\epsilon_2^\alpha - \sqrt{2}\frac{\left(\frac{\epsilon_1^\alpha}{\epsilon_2^\alpha}\right)'}{1+2\left(\frac{\epsilon_1^\alpha}{\epsilon_2^\alpha}\right)^2}}{\frac{1}{\sqrt{2}}\sqrt{2\epsilon_1^{\alpha^2} + \epsilon_2^{\alpha^2}}}.$$

Since α is assumed to be a slant helix, we have $g' = 0$, where $g = \frac{\epsilon_1^\alpha}{\epsilon_2^\alpha}$.

Substituting this condition simplifies the expression to

$$\frac{\tau_\beta}{\kappa_\beta} = -\frac{1}{\sqrt{1+2g^2}}.$$

Since this expression is constant, it follows that the $\mathbf{M}_1^\alpha\mathbf{B}_\alpha$-type-2 Bishop adjoint curve of α is a general helix.

Corollary 3.2.5. Let the unit speed curve $\alpha: I \to E^3$ be a slant helix with the Bishop frame. If we rotate the Frenet frame around the $\mathbf{B}_\alpha$ axis, the angle of rotation is

$$\theta_\beta = \frac{1}{\sqrt{2}}\int_0^s \sqrt{2\epsilon_1^{\alpha^2} + \epsilon_2^{\alpha^2}}\,ds.$$

Proof 3.2.5. According to type-2 Bishop formulas, the angle function θ_β is given by:

$$\theta_\beta = \int_0^s \kappa(s)ds.$$

Substituting the expression for κ_β, we obtain:

$$\theta_\beta = \frac{1}{\sqrt{2}}\int_0^s \sqrt{2\epsilon_1^{\alpha^2} + \epsilon_2^{\alpha^2}}\,ds.$$

3.3. $\mathbf{M}_2^\alpha\mathbf{B}_\alpha$-type-2 Bishop adjoint curve

Theorem 3.3.1. Let α be an *s*-arc length parameterized regular curve in E^3 with Bishop apparatus $\{\mathbf{M}_1^\alpha, \mathbf{M}_2^\alpha, \mathbf{B}_\alpha, \epsilon_1^\alpha, \epsilon_2^\alpha\}$. The Frenet vector fields, curvature and torsion of γ are given by

$$\mathbf{T}_\gamma = \frac{1}{\sqrt{2}}(\mathbf{M}_2^\alpha + \mathbf{B}_\alpha), \tag{15}$$

$$\mathbf{N}_\gamma = \frac{1}{\sqrt{1+2g^2}}(\mathbf{M}_1^\alpha + g\mathbf{M}_2^\alpha - g\mathbf{B}_\alpha), \tag{16}$$

$$\mathbf{B}_\gamma = \frac{1}{\sqrt{2+4g^2}}(-2g\mathbf{M}_1^\alpha + \mathbf{M}_2^\alpha - \mathbf{B}_\alpha), \tag{17}$$

$$\kappa_\gamma = \frac{1}{\sqrt{2}}\epsilon_2^\alpha\sqrt{1+2g^2} \tag{18}$$

$$\tau_\gamma = -\frac{1}{2\sqrt{2}}\epsilon_1^\alpha - \frac{g'}{2+g^2}, \tag{19}$$

where $g = \frac{\epsilon_1^\alpha}{\epsilon_2^\alpha}$.

Proof 3.3.1. Differentiating Eq (7) and applying the Frenet formulas, we obtain

$$\frac{d\gamma}{d\tilde{s}}\frac{d\tilde{s}}{ds} = \frac{1}{\sqrt{2}}(\mathbf{M}_{\mathbf{2}}^{\boldsymbol{\alpha}} + \mathbf{B}_{\boldsymbol{\alpha}}),$$

$$\mathbf{T}_{\gamma}\frac{d\tilde{s}}{ds} = \frac{1}{\sqrt{2}}(\mathbf{M}_{\mathbf{2}}^{\boldsymbol{\alpha}} + \mathbf{B}_{\boldsymbol{\alpha}}),$$

taking the norm on both sides gives:

$$\frac{d\tilde{s}}{ds} = 1,$$

Thus, we conclude:

$$\mathbf{T}_{\gamma} = \frac{1}{\sqrt{2}}(\mathbf{M}_{\mathbf{2}}^{\boldsymbol{\alpha}} + \mathbf{B}_{\boldsymbol{\alpha}})$$

differentiating (15) with respect to s and using Eq (2) we find

$$\mathbf{T}_{\gamma}' = \frac{1}{\sqrt{2}}(\epsilon_1^{\alpha}\mathbf{M}_{\mathbf{1}}^{\boldsymbol{\alpha}} + \epsilon_2^{\alpha}\mathbf{M}_{\mathbf{2}}^{\boldsymbol{\alpha}} - \epsilon_2^{\alpha}\mathbf{B}_{\boldsymbol{\alpha}}).$$

From this, the curvature κ_{γ} and principal normal vector $\mathbf{N}_{\gamma}$ of curve γ are given by:

$$\kappa_{\gamma} = \left\|\mathbf{T}_{\gamma}'\right\| = \frac{1}{\sqrt{2}}\sqrt{\epsilon_1^{\alpha^2} + 2\epsilon_2^{\alpha^2}},$$

$$\mathbf{N}_{\gamma} = \frac{1}{\sqrt{\epsilon_1^{\alpha^2} + 2\epsilon_2^{\alpha^2}}}(\epsilon_1^{\alpha}\mathbf{M}_{\mathbf{1}}^{\boldsymbol{\alpha}} + \boldsymbol{\epsilon}_{\mathbf{2}}^{\boldsymbol{\alpha}}\mathbf{M}_{\mathbf{2}}^{\boldsymbol{\alpha}} - \boldsymbol{\epsilon}_{\mathbf{2}}^{\boldsymbol{\alpha}}\mathbf{B}_{\boldsymbol{\alpha}}).$$

Next, the binormal vector is determined as:

$$\mathbf{B}_{\gamma} = \mathbf{T}_{\gamma} \times \mathbf{N}_{\gamma}.$$

Substituting their values:

$$\mathbf{B}_{\gamma} = \frac{1}{\sqrt{2\epsilon_1^{\alpha^2} + 4\epsilon_2^{\alpha^2}}}(-2\epsilon_2^{\alpha}\mathbf{M}_{\mathbf{1}}^{\boldsymbol{\alpha}} + \epsilon_1^{\alpha}\mathbf{M}_{\mathbf{2}}^{\boldsymbol{\alpha}} - \epsilon_1^{\alpha}\mathbf{B}_{\boldsymbol{\alpha}}).$$

To find the torsion τ_{γ}, we differentiate $\mathbf{N}_{\gamma}$ and use:

$$\tau_{\gamma} = \langle \mathbf{N}_{\gamma}', \mathbf{B}_{\gamma} \rangle.$$

After simplification, we obtain:

$$\tau_\gamma = \frac{1}{2\sqrt{2}}\epsilon_1^\alpha - \frac{\left(\frac{\epsilon_1^\alpha}{\epsilon_2^\alpha}\right)'}{1 + 2\left(\frac{\epsilon_1^\alpha}{\epsilon_2^\alpha}\right)^2}$$

Setting:

$$g = \frac{\epsilon_1^\alpha}{\epsilon_2^\alpha}$$

we arrive at the final result.

Theorem 3.3.2. Let α be an *s*-arc length parameterized regular curve in E^3 ,equipped with the type-2 Bishop apparatus $\{M_1^\alpha, M_2^\alpha, B_\alpha, \epsilon_1^\alpha, \epsilon_2^\alpha\}$. Let γ be the $\mathbf{M}_2^\alpha\mathbf{B}_\alpha$-type-2 Bishop adjoint curve of α. The type-2 Bishop vector fields and curvatures associated with γ are given by:

$$N_1^\gamma = \frac{\cos\theta_\gamma}{\sqrt{1+2g^2}}\mathbf{M}_1^\alpha + \left(\frac{1}{\sqrt{2}}\sin\theta_\gamma + \frac{g}{\sqrt{1+2g^2}}\cos\theta_\gamma\right)\mathbf{M}_2^\alpha + \ + \left(\frac{1}{\sqrt{2}}\sin\theta_\gamma - \frac{g}{\sqrt{1+2g^2}}\cos\theta_\gamma\right)\mathbf{B}_\alpha$$

$$N_2^\gamma = \frac{\sin\theta_\gamma}{\sqrt{1+2g^2}}\mathbf{M}_1^\alpha + \left(-\frac{1}{\sqrt{2}}\cos\theta_\gamma + \frac{g}{\sqrt{1+2g^2}}\sin\theta_\gamma\right)\mathbf{M}_2^\alpha + \left(\frac{1}{\sqrt{2}}\cos\theta_\gamma - \frac{g}{\sqrt{1+2g^2}}\sin\theta_\gamma\right)\mathbf{B}_\alpha$$

$$\epsilon_1^\gamma = \left(\frac{1}{2\sqrt{2}}\epsilon_1^\alpha + \frac{g'}{2+g^2}\right)\cos\theta_\gamma \ , \quad \epsilon_2^\gamma = \left(\frac{1}{2\sqrt{2}}\epsilon_1^\alpha + \frac{g'}{2+g^2}\right)\sin\theta_\gamma.$$

Proof 3.3.2. By substituting Eqs. (15) and (16) into Eqs (4) and (5), the proof follows directly.

Corollary 3.3.3. If the unit speed curve $\alpha: I \to E^3$ be a slant helix with respect to the Bishop frame, then its $\mathbf{M}_2^\alpha\mathbf{B}_\alpha$-type-2 Bishop adjoint curve is a general helix.

Proof 3.3.3. By computing the the ratio of the torsion and curvature of the $\mathbf{M}_2^\alpha\mathbf{B}_\alpha$ -Bishop adjoint curve of α , as given in Theorem **3.3.1.**, we obtain:

$$\frac{\tau_\gamma}{\kappa_\gamma}=\frac{\frac{1}{2\sqrt{2}}\epsilon_1^\alpha-\frac{\left(\frac{\epsilon_1^\alpha}{\epsilon_2^\alpha}\right)'}{1+2\left(\frac{\epsilon_1^\alpha}{\epsilon_2^\alpha}\right)^2}}{\frac{1}{\sqrt{2}}\sqrt{\epsilon_1^{\alpha^2}+2\epsilon_2^{\alpha^2}}}.$$

Since α is assumed to be a slant helix, we have $g'=0$, where $g=\frac{\epsilon_1^\alpha}{\epsilon_2^\alpha}$.

Substituting this condition simplifies the expression to

$$\frac{\tau_\gamma}{\kappa_\gamma}=-\frac{1}{\sqrt{1+2g^2}}.$$

Since this expression is constant, it follows that the $\mathbf{M}_2^\alpha\mathbf{B}_\alpha$ type-2 Bishop adjoint curve of α is a general helix.

Corollary 3.3.4. Let the unit speed curve $\alpha: I \to E^3$ be a slant helix with the Bishop frame. If we rotate the Frenet frame around the $\mathbf{B}_\alpha$ axis, the angle of rotation is

$$\theta_\gamma=\frac{1}{\sqrt{2}}\int_0^s\sqrt{\epsilon_1^{\alpha^2}+2\epsilon_2^{\alpha^2}}.$$

Proof 3.3.4. According to type-2 Bishop formulas, the angle function θ_γ is given by:

$$\theta_\gamma=\int_0^s\kappa(s)ds.$$

Substituting the expression for κ_γ , we obtain:

$$\theta_\gamma=\frac{1}{\sqrt{2}}\int_0^s\sqrt{\epsilon_1^{\alpha^2}+2\epsilon_2^{\alpha^2}}\,ds.$$

3.4. $\mathbf{M}_1^\alpha\mathbf{M}_2^\alpha$-type-2 Bishop adjoint curve

Theorem 3.4.1 Let α be an *s*-arc length parameterized regular curve in E^3 with Bishop apparatus $\{\mathbf{M}_1^\alpha,\mathbf{M}_2^\alpha,\mathbf{B}_\alpha,\epsilon_1^\alpha,\epsilon_2^\alpha\}$. The Frenet vector fields, curvature and torsion of Ω are given by

$$\mathbf{T}_\Omega = \frac{1}{\sqrt{2}}(\mathbf{M}_1^\alpha + \mathbf{M}_2^\alpha), \tag{20}$$

$$\mathbf{N}_\Omega = -\mathbf{B}_\alpha, \tag{21}$$

$$\mathbf{B}_\Omega = \frac{1}{\sqrt{2}}(-\mathbf{M}_1^\alpha + \mathbf{M}_2^\alpha), \tag{22}$$

$$\kappa_\Omega = \frac{1}{\sqrt{2}}(\epsilon_1^\alpha + \epsilon_2^\alpha) \tag{23}$$

$$\tau_\Omega = 0 \tag{24}$$

where $g = \frac{\epsilon_1^\alpha}{\epsilon_2^\alpha}$.

Proof 3.4.1. Differentiating Eq (8) and applying the Frenet formulas, we obtain

$$\frac{d\Omega}{d\tilde{s}}\frac{d\tilde{s}}{ds} = \frac{1}{\sqrt{2}}(\mathbf{M}_1^\alpha + \mathbf{M}_2^\alpha),$$

$$\mathbf{T}_\Omega\frac{d\tilde{s}}{ds} = \frac{1}{\sqrt{2}}(\mathbf{M}_1^\alpha + \mathbf{M}_2^\alpha),$$

taking the norm on both sides gives:

$$\frac{d\tilde{s}}{ds} = 1,$$

thus, we conclude:

$$\mathbf{T}_\Omega = \frac{1}{\sqrt{2}}(\mathbf{M}_1^\alpha + \mathbf{M}_2^\alpha)$$

differentiating (20) with respect to s and using Eq (2) we find

$$\mathbf{T}_\Omega' = -\frac{1}{\sqrt{2}}(\epsilon_1^\alpha + \epsilon_2^\alpha)\mathbf{B}_\alpha.$$

From this, the curvature κ_Ω and principal normal vector $\mathbf{N}_\Omega$ of curve Ω are given by:

$$\kappa_\Omega = \|\mathbf{T}_\Omega'\| = \frac{1}{\sqrt{2}}(\epsilon_1^\alpha + \epsilon_2^\alpha),$$

$$\mathrm{N}_\Omega = -\mathbf{B}_\alpha.$$

Next, the binormal vector is determined as:

$$\mathbf{B}_\Omega = \mathbf{T}_\Omega \times \mathrm{N}_\Omega$$

Substituting their values:

$$\mathbf{B}_\Omega = \frac{1}{\sqrt{2}}(-\mathbf{M}_1^\alpha + \mathbf{M}_2^\alpha),$$

to find the torsion τ_Ω, we differentiate $\mathbf{N}_\Omega$ and use:

$$\tau_\Omega = \langle \mathbf{N}'_\Omega, \mathbf{B}_\Omega \rangle.$$

After simplification, we obtain

$$\tau_\Omega = 0$$

we arrive at the final result.

Corollary 3.4.2. For the unit speed curve $\alpha: \mathrm{I} \to \mathrm{E}^3$ its $\mathbf{M}_1^\alpha \mathbf{M}_2^\alpha$ -type-2 Bishop adjoint curve is a planar curve.

Proof 3.4.2. The proof is clear from Equation (24).

Corollary 3.4.3. For the unit speed curve $\alpha: \mathrm{I} \to \mathrm{E}^3$ its $\mathbf{M}_1^\alpha \mathbf{M}_2^\alpha$ -type-2 Bishop adjoint curve is a Mannheim partner.

Proof 3.4.3. The proof is clear from Equation (21).

Corollary 3.4.4. Let the unit speed curve $\alpha: \mathrm{I} \to \mathrm{E}^3$ be a slant helix with the Bishop frame. If we rotate the Frenet frame around the $\mathbf{B}_\alpha$ axis, the angle of rotation is

$$\theta_\Omega = \frac{1}{\sqrt{2}} \int_0^s (\epsilon_1^\alpha + \epsilon_2^\alpha).$$

Proof 3.4.4. According to type-2 Bishop formulas, the angle function θ_Ω is given by:

$$\theta_\Omega = \int_0^s \kappa(s) ds.$$

Substituting the expression for κ_Ω, we obtain:

$$\theta_\Omega = \frac{1}{\sqrt{2}} \int_0^s (\epsilon_1^\alpha + \epsilon_2^\alpha)\, ds.$$

4. RESULTS

In this study, new adjoint curves were introduced by combining special Smarandache curves with integral curves in the type-2 Bishop frame. The geometric properties of these adjoint curves were systematically examined, and relationships between the original curve and its adjoint curves were established. As a result, necessary and sufficient conditions were derived for a curve to be classified as a general helix or a slant helix.

Furthermore, it was demonstrated that certain adjoint curves correspond to well-known curve types, such as Mannheim partners or planar curves, under specific conditions. These findings contribute to the broader understanding of differential geometry by offering new perspectives on curve classifications and their interactions within the type-2 Bishop frame. The results obtained in this study can have applications in physics, engineering, and computer-aided geometric design, where curve structures play a crucial role. Future research may extend these adjoint curve concepts to different geometric spaces and explore their implications in various applied fields.

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CHAPTER 3

APPROXIMATION FOR INTEGRO-DIFFERENTIAL EQUATIONS WITH THREE-STEP ITERATION METHOD

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Introduction

Fractional analysis is a mathematical discipline that focuses on derivatives and integrals of real or complex order. Differential equations with non-integer derivatives serve as essential tools for modeling various physical phenomena. Consequently, beyond its mathematical significance, fractional calculus finds applications in diverse scientific fields, including physics, engineering, biology, and finance (see [1]-[4]). Some of the most extensive studies on fractional derivatives and integrals can be found in (see [5], [6]).

One of the most intriguing aspects of fractional calculus is its reliance on operators. This characteristic allows researchers to select the most suitable operator for describing dynamic behaviors in real-world problems. The necessity of identifying optimal operators for different applications has led to the discovery of novel fractional operators, continuously enhancing the accuracy of models representing natural phenomena (see [7]-[9]). Additionally, various other types of innovative fractional derivatives have been explored in the literature (see [10], [11]).

Meanwhile, fixed point theory stands as one of the most powerful tools in nonlinear analysis, particularly in solving differential, integral, and partial differential equations. It plays a fundamental role in numerous mathematical applications. Fixed point theorems provide significant insights into the existence and uniqueness of solutions for fractional differential equations, particularly in initial value and boundary value problems. However, obtaining exact analytical solutions for these equations is often challenging. As a result, extensive research has been conducted on numerical and approximation methods for solving fractional differential equations. In line with these efforts, various studies have focused on proving the existence and uniqueness of solutions (see [12]-[15] and references therein).

Building on fixed point theory, numerous contraction-type transformations have been introduced. These include Lipschitzian transformations, contraction transformations, contraction-like transformations, non-expanding transformations, pseudo-contractions, semi-contractions, and weak contractions, among others (see [16]-[19]).

In parallel with these advancements, various iteration methods have been developed and widely studied in fixed point theory. Some of the most notable iterative techniques include the Mann iteration method (see [20]), Krasnosel'skii iteration method (see [21]), Kirk iteration method (see [22]), Ishikawa iteration method (see [18]), Noor iteration method (see [23]), S iteration method (see [24]), and the recently introduced three-step iteration method (see [19]). Additionally, hybrid iteration methods that combine these approaches have proven to be highly effective. Some of these hybrid methods include the Kirk-Noor iteration method, Kirk-

Ishikawa and Kirk-Mann iteration methods, Picard-Mann iteration method, Mann-Picard method, and Kirk-MP iteration method (see [25]-[28]).

Furthermore, in an iteration algorithm constructed using a given transformation, an alternative transformation known as the approximation operator can also be applied. Since the fixed point of the approximation operator differs from that of the original transformation, discrepancies may arise between the two. The concept of data dependency, which has been extensively examined in the literature, addresses the extent of this difference and how it can be quantified (see [34]-[37]).

The structure of the paper is as follows: In Section 2, we present fundamental definitions and key theoretical results essential for our analysis. Section 3, we consider the following initial value problem for integro-differential equation, $f \in (C[0,A] \times \mathbb{R}, \mathbb{R})$

$$\begin{cases} \frac{d^2u(s)}{ds^2} + I_{a^+}^{\alpha}\, u(s) = f\big(s\,, u(s), u'(s)\big) \\ \quad 0 < s < A\ ,\ 0 < \alpha < 1 \\ \quad u(s) = 0\,,\ \ u'(s) = 0 \end{cases} \tag{1}$$

where A constant number and $f \in (C[0,A] \times \mathbb{R}^2, \mathbb{R})$. Firstly, strong convergence of integro-differential equation is investigated by using the three-step iteration algorithm defined by Karakaya et al ([19]). Also, data dependency is obtained for integro-differential equations. Finally, we conclude with a discussion of our findings and potential future research directions.

2. Known Results

We present some basic definitions and preliminary facts which are used through the paper.

Definition 2.1: *Let $u(t) \in C([a,b])$ and $a < t < b$, $\alpha \in (-\infty, \infty)$. The Riemann-Liouville fractional integral of order α is defined by*

$$I_{a+}^{\alpha}u(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{u(s)}{(t-s)^{1-\alpha}}\, ds.$$

The same definition for $\alpha \in (0,1)$ can be expressed as

$$D_{a+}^{\alpha}u(t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{u(s)}{(t-s)^{\alpha}}\, ds.$$

The Riemann-Liouville fractional derivative of order α. Here,

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt \ , \quad (\alpha > 0).$$

(see [6]).

Definition 2.2: $C^{(1)}[a,b] = (C^{(1)}[a,b], d)$ *is the complete space defined on the interval* $[a,b]$ *with the metric* d *defined by*

$$d(u,v) = \max_{s\in[0,A]} |u(s) - v(s)| + \max_{s\in[0,A]} |u'(s) - v'(s)|$$

(see [33]).

Definition 2.3: *Let* (E,d) *be a metric space and* $T: E \to E$ *be a mapping.* T *is called a Lipschitzian mapping, if there is a number* $L > 0$ *such that* $d(Tu,Tv) \leq Ld(u,v)$ *for all* $u,v \in E$ *(see [29]).*

Definition 2.4: *Let* (E,d) *be a metric space and* $T: E \to E$ *be a Lipschitzian mapping. T is called a contraction mapping, if there is at least one* $\lambda \in (0,1)$ *real number such that* $d(Tu,Tv) \leq \lambda d(u,v)$ *for all* $u,v \in E$. λ *is called the contraction ratio (see [29]).*

The definition of contraction mapping in E normed space can be expressed as follows:

$$\|Tu - Tv\| \leq \lambda \|u - v\|$$

where $\lambda \in (0,1)$ the contraction ratio (see [29]).

Now, let's state the Banach fixed point theorem (see [30]).

Theorem 2.1: *If* (E,d) *is a complete metric space and* $T: E \to E$ *is a contraction mapping,*

i. *T has one and only one fixed point* $u \in E$.

ii. *For any* $u_0 \in E$*, iteration sequence* $(T^n u_0)$ *(ie iteration sequence* (u_n) *defined by* $u_n = Tu_{n-1}$ *for all* $n \in \mathbb{N}$*) converges to unique fixed point of* T.

The following three steps iteration algorithm, defined by Karakaya et al. in 2017 (see [19]), has been shown to be faster than many iteration algorithms such as Picard, Mann, Ishikawa, Noor, SP, S, CR and Picard-S:

Definition 2.5: *The iteration method*

$$\begin{cases} u_{n+1} = Tv_n \\ v_n = (1-\beta_n)w_n + \beta_n T w_n \\ w_n = Tu_n \end{cases} \tag{2}$$

for $u_0 \in E$, *where* E *is a Banach space,* $T: E \to E$ *is an operator, and* $\{\beta_n\}_{n=0}^{\infty} \subset [0,1]$ *is a sequence satisfying certain conditions, is called the three-step iteration method.*

Theorem 2.2: *Let* f *be a continuous function defined on*

$$D = \{(s,u,v):\ s \in [0,A]\} \subseteq \mathbb{R}^3,$$

and there exists a constant $k > 0$ *such that* $|f(s,u,v)| \leq k$ *for all* $s \in D$. *And, suppose that* f *satisfies a Lipschitz condition on D with respect to its second and third arguments. Thus, for arbitrary* $(s,u,z),(s,v,w) \in D$ *there is a positive constant L such that*

$$|f(s,u,z) - f(s,v,w)| \leq L(|u - v| + |z - w|) \tag{3}$$

is valid. Also, let

$$h(\alpha, A, L) = \frac{A^{\alpha+2}}{\Gamma(\alpha+3)} + L\frac{A^2}{2},$$

and suppose that

$$h(\alpha, A, L) < 1. \tag{4}$$

Then, initial value problem (1) has a unique solution $u \in C^{(1)}[0,A]$.

Definition 2.6: *Let* $T_1, T_2: K \to K$ *be operator. If* $\|T_1 u - T_2 u\| \leq \varepsilon$ *for each* $u \in K$ *and constant* $\varepsilon > 0$, *then* T_2 *is called the approximation operator of* T_1 *(see [31]).*

Lemma 2.1: *Let* $\{a_n\}_{n=0}^{\infty}$ *be a non-negative real sequence and there exists* $n_0 \in \mathbb{N}$ *such that for each* $n \geq n_0$ *satisfting the following condition:*

$$a_{n+1} \leq (1-\mu_n)a_n + \mu_n \gamma_n,$$

where $\mu_n \in (0,1)$ *such that* $\sum_{n=0}^{\infty} \mu_n = \infty$ *and* $\gamma_n \geq 0$. *Then the following inequality holds:*

$$0 \leq \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} \gamma_n$$

(see [31]).

3 Main Result

Theorem 3.1: *Let* $T: \left(C^{(1)}[0,A], \|\cdot\|\right) \to \left(C^{(1)}[0,A], \|\cdot\|\right)$ *be an operator and* $\{\beta_n\}_{n=0}^{\infty} \subset [0,1]$ *be a sequence satisfying certain conditions. In this case, the integro-differential equation given by equation (1) has a unique solution in the form of* $u^* \in C[0,A]$ *and the sequence* $\{u_n\}_{n=0}^{\infty}$ *obtained from the iteration algorithm given by equation (2) converges to this solution.*

Proof: By integrating both sides of integro-differential equation (1), we obtain integral equation

$$u(s) = -\frac{1}{\Gamma(\alpha)}\int_0^s \int_0^r \left(\int_0^q (q-t)^{\alpha-1} u(t)dt\right)dqdr + \int_0^s \int_0^r f\left(q\,, u(q), u'(q)\right)dqdr. \qquad (5)$$

We can be written in the equivalent integral form (5), which is in the form $u = Tu$, where $T: C^{(1)}[0,A] \to C^{(1)}[0,A]$ is an operator defined by

$$Tu(s) = -\frac{1}{\Gamma(\alpha)}\int_0^s \int_0^r \left(\int_0^q (q-t)^{\alpha-1} u(t)dt\right)dqdr + \int_0^s \int_0^r f\left(q, u(q), u'(q)\right)dqdr \qquad (6)$$

where f is continuous function on the rectangle D. Consider the sequence $\{u_n\}_{n=0}^{\infty}$ obtained from the iteration algorithm given by equation (2) constructed with the operator $T: \left(C^{(1)}[0,A], \|\cdot\|\right) \to \left(C^{(1)}[0,A], \|\cdot\|\right)$. It will be shown that for $n \to \infty$ is $u_n \to u_*$. Using equation (2) and conditions of Theorem 2.2, we are obtained the following inequality:

$$|u_{n+1}(t) - u_*(t)| = |Tv_n(t) - Tu_*(t)|$$

$$= \Bigg| -\frac{1}{\Gamma(\alpha)}\int_0^s \int_0^r \left(\int_0^q (q-t)^{\alpha-1} v_n(t)dt\right)dqdr + \int_0^s \int_0^r f\left(q\,, v_n(q), v_n'(q)\right)dqdr$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^s \int_0^r \left(\int_0^q (q-t)^{\alpha-1} u_*(t)dt\right)dqdr - \int_0^s \int_0^r f\left(q\,, u_*(q), u_*'(q)\right)dqdr\Bigg|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^s \int_0^r \int_0^q (q-t)^{\alpha-1} \left|v_n(t) - u_*(t)\right| dtdqdr$$

$$+\int_0^s \int_0^r \left|f\left(q\,, v_n(q), v_n'(q)\right) - f\left(q\,, u_*(q), u_*'(q)\right)\right| dqdr$$

$$\leq \frac{1}{\Gamma(\alpha)}\|v_n - u^*\|_\infty \int_0^s \int_0^r \int_0^q (q-t)^{\alpha-1}\, dtdqdr$$
$$+ \int_0^s \int_0^r L(|\, v_n(q) - \ u_*(q)| + |v_n'(q) - u_*'(q)|)dqdr$$
$$\leq \left(\frac{A^{\alpha+2}}{\Gamma(\alpha+3)} + L\frac{A^2}{2}\right)\|v_n - u^*\|.$$

Since $h(\alpha, A, L) = \frac{A^{\alpha+2}}{\Gamma(\alpha+3)} + L\frac{A^2}{2}$, we have

$$\|u_{n+1} - u_*\| \leq h\|v_n - u_*\|. \qquad (7)$$

By making the necessary calculations, the following inequalities are obtained

$$|v_n(t) - u_*(t)| = |(1-\beta_n)w_n(t) + \beta_n T w_n(t) - Tu_*(t)|$$
$$\leq (1-\beta_n)\|w_n - u_*\| + \beta_n\|Tw_n - Tu_*\|$$
$$\leq (1-\beta_n)\|w_n - u_*\|$$
$$+\beta_n \left\| \begin{matrix} -\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^r\left(\int_0^q (q-t)^{\alpha-1} w_n(t)dt\right)dqdr + \int_0^s\int_0^r f(q\,, w_n(q), w_n'(q))dqdr \\ -\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^r\left(\int_0^q (q-t)^{\alpha-1} u_*(t)dt\right)dqdr + \int_0^s\int_0^r f(q\,, u_*(q), u_*'(q))dqdr \end{matrix} \right\|$$
$$\leq (1-\beta_n)\|w_n - u_*\|$$
$$+\beta_n\left\|\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^r\int_0^q (q-t)^{\alpha-1}\big(w_n(t) - u_*(t)\big)\, dtdqdr\right\|$$
$$+\beta_n\left\|\int_0^s\int_0^r \Big(f\big(q\,, w_n(q), w_n'(q)\big) - f\big(q\,, u_*(q), u_*'(q)\big)\Big)\, dqdr\right\|$$
$$\leq (1-\beta_n)\|w_n - u_*\|$$
$$+\frac{1}{\Gamma(\alpha)}\beta_n\int_0^s\int_0^r\int_0^q \|w_n - u_*\|\, dtdqdr + \beta_n\int_0^s\int_0^r \|f(q\,, w_n, w_n') - f(q\,, u_*, u_*')\|\, dqdr$$

$$= (1-\beta_n)\|w_n - u_*\|$$
$$+\beta_n\left(\frac{A^{\alpha+2}}{\Gamma(\alpha+3)}\right)\|w_n - u_*\| + \beta_n L\frac{A^2}{2}\|w_n - u_*\|$$

$$= (1-\beta_n)\|w_n - u_*\| + \left(\frac{A^{\alpha+2}}{\Gamma(\alpha+3)} + L\frac{A^2}{2}\right)\beta_n\|w_n - u_*\|$$
$$= (1-\beta_n)\|w_n - u_*\| + h\beta_n\|w_n - u_*\|$$
$$= (1-\beta_n + h\beta_n)\|w_n - u_*\|.$$

So,

$$\|v_n - u_*\| \le (1-\beta_n + h\beta_n)\|w_n - u_*\|. \qquad (8)$$

Similarly,

$$|w_n(t) - u_*(t)| = |Tu_n(t) - Tu_*(t)|$$
$$= \left| -\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^r\left(\int_0^q (q-t)^{\alpha-1}u_n(t)dt\right)dqdr + \int_0^s\int_0^r f(q, u_n(q), u_n'(q))dqdr \right.$$
$$\left. + \frac{1}{\Gamma(\alpha)}\int_0^s\int_0^r\left(\int_0^q (q-t)^{\alpha-1}u_*(t)dt\right)dqdr - \int_0^s\int_0^r f(q, u_*(q), u_*'(q))dqdr \right|$$
$$\le \frac{1}{\Gamma(\alpha)}\int_0^s\int_0^r\int_0^q \|u_n - u_*\|\, dtdqdr + \int_0^s\int_0^r \|f(q, u_n, u_n') - f(q, u_*, u_*')\|\, dqdr$$
$$= \left(\frac{A^{\alpha+2}}{\Gamma(\alpha+3)} + L\frac{A^2}{2}\right)\|u_n - u_*\|$$
$$= h\|u_n - u_*\|$$

$$\|w_n - u_*\| \le h\|u_n - u_*\| \qquad (9)$$

is found.

If inequalities (9) and (8) are written in inequality (7), the following inequality is obtained.

$$\|u_{n+1} - u_*\| \le h^2(1-\beta_n + h\beta_n)\|u_n - u_*\|$$
$$\|u_n - u_*\| \le h^2(1-\beta_n + h\beta_n)\|u_{n-1} - u_*\|$$
$$\vdots$$
$$\|u_1 - u_*\| \le h^2(1-\beta_n + h\beta_n)\|u_0 - u_*\|$$

We are obtain the following inequality by applying induction to the last inequality,

$$\|u_{n+1} - u_*\| \le h^{2(n+1)}\prod_{i=0}^{n}[1-\beta_i + h\beta_i]\,\|u_0 - u_*\|.$$

Also, $\forall x \in [0,1]$ for we have $1 - x \leq e^{-x}$. So,

$$\|u_{n+1} - u_*\| \leq h^{2(n+1)} \|u_0 - u_*\| \prod_{i=0}^{n} e^{-\beta_i + h\beta_i}$$
$$\|u_{n+1} - u_*\| \leq h^{2(n+1)} \|u_0 - u_*\| e^{-(1-h)\sum_{i=0}^{n} \beta_i},$$

we obtain

$$\lim_{n\to\infty} \|u_n - u_*\| = 0.$$

Now, let us examine the data dependency of the solution of the integro-differential equation given by equation (1) using the iteration algorithm given in equation (2). Thus, we consider the following initial value problem for second integro-differential equation

$$\begin{cases} \dfrac{d^2\varphi(s)}{ds^2} + I_{a^+}^{\alpha}\, \varphi(s) = g\big(s, x(s), x'(s)\big) \\ \qquad 0 < s < A\ ,\ \ 0 < \alpha < 1 \\ \qquad \varphi(s) = 0\ ,\ \ \varphi'(s) = 0\ . \end{cases} \tag{10}$$

By integrating both sides of integro-differential equation (10), we obtain integral equation

$$\varphi(t) = -\frac{1}{\Gamma(\alpha)} \int_0^s \int_0^r \Big(\int_0^q (q-t)^{\alpha-1} x(t)dt\Big) dqdr + \int_0^s \int_0^r g\big(q, x(q), x'(q)\big) dqdr \tag{11}$$

We can be written in the equivalent integral form (11), which is in the form $x = Sx,$ where $S: (C^{(1)}[0,A], \|\cdot\|) \to (C^{(1)}[0,A], \|\cdot\|)$ is an operator defined by

$$Sx(t) = -\frac{1}{\Gamma(\alpha)} \int_0^s \int_0^r \Big(\int_0^q (q-t)^{\alpha-1} x(t)dt\Big) dqdr + \int_0^s \int_0^r g\big(q, x(q), x'(q)\big) dqdr \tag{12}$$

where g is continuous function on the rectangle D.
If the iteration algorithm given in equation (2) is reconstructed with operators $T(6)$ and $S(12)$, respectively, the following iteration algorithms can be written.

$$\begin{cases} u_{n+1}(t) = -\frac{1}{\Gamma(\alpha)} \int_0^s \int_0^r \Big(\int_0^q (q-t)^{\alpha-1} v_n(t)dt\Big) dqdr + \int_0^s \int_0^r f\big(q\ , v_n(q), v_n'(q)\big) dqdr \\ v_n(t) = -\frac{1}{\Gamma(\alpha)} \int_0^s \int_0^r \Big(\int_0^q (q-t)^{\alpha-1} w_n(t)dt\Big) dqdr + \int_0^s \int_0^r f\big(q\ , w_n(q), w_n'(q)\big) dqdr \\ w_n(t) = -\frac{1}{\Gamma(\alpha)} \int_0^s \int_0^r \Big(\int_0^q (q-t)^{\alpha-1} u_n(t)dt\Big) dqdr + \int_0^s \int_0^r f\big(q\ , u_n(q), u_n'(q)\big) dqdr \end{cases} \tag{13}$$

$$\begin{cases}\varphi_{n+1}(t) = -\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^r\left(\int_0^q(q-t)^{\alpha-1}\psi_n(t)dt\right)dqdr + \int_0^s\int_0^r g\left(q\,,\psi_n(q),\psi_n'(q)\right)dqdr \\ \psi_n(t) = -\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^r\left(\int_0^q(q-t)^{\alpha-1}\phi_n(t)dt\right)dqdr + \int_0^s\int_0^r g\left(q\,,\phi_n(q),\phi_n'(q)\right)dqdr \\ \phi_n(t) = -\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^r\left(\int_0^q(q-t)^{\alpha-1}\varphi_n(t)dt\right)dqdr + \int_0^s\int_0^r g\left(q\,,\varphi_n(q),\varphi_n'(q)\right)dqdr\end{cases} \tag{14}$$

Theorem 3.2: *Let the sequence* $\{\beta_n\}_{n=0}^{\infty} \subset [0,1]$ *satisfy the condition* $\beta_n \geq \frac{1}{2}$ *for each* $n \in \mathbb{N}$. *Consider the sequence* $\{u_n\}_{n=0}^{\infty}$ *obtained from equation (13) and the sequence* $\{\varphi_n\}_{n=0}^{\infty}$ *obtained from equation (14). Let the solutions of the integral equations (5) and (11) be* u_* *and* p, *respectively, with the conditions of Theorem 3.1*

i) *Let the constant* ε_1 *exist such that*

$$\left\|f\left(q\,,u_n(q),u_n'(q)\right) - g\left(q\,,\varphi_n(q),\varphi_n'(q)\right)\right\| \leq \varepsilon_1$$

for each $(t,s) \in [0,A]$.

ii) *Let* $M = \frac{A^{\alpha+1}}{\Gamma(\alpha+3)} \leq 1$.

If $u_n \to u_*$ *and* $\varphi_n \to p$ *as* $n \to \infty$, *then the inequality*

$$\|u_* - p\| \leq \frac{5A^2\varepsilon_1}{2(1-M)}$$

is valid.

Proof: With the hypotheses of the Theorem 3.2, the following inequalities (15), (16) and (17) are obtained.

$$\|u_{n+1} - \varphi_{n+1}\| = \|Tv_n - S\psi_n\|$$

$$= \left\|\begin{matrix} -\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^r\left(\int_0^q(q-t)^{\alpha-1}v_n(t)dt\right)dqdr + \int_0^s\int_0^r f\left(q,v_n(q),v_n'(q)\right)dqdr \\ -\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^r\left(\int_0^q(q-t)^{\alpha-1}\psi_n(t)dt\right)dqdr + \int_0^s\int_0^r g\left(q,\psi_n(q),\psi_n'(q)\right)dqdr \end{matrix}\right\|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^s\int_0^r\int_0^q(q-t)^{\alpha-1}\,\|v_n - \psi_n\|dtdqdr$$

$$+\int_0^s\int_0^r\left\|f\left(q\,,v_n(q),v_n'(q)\right) - g\left(q\,,\psi_n(q),\psi_n'(q)\right)\right\|\,dqdr$$

$$\leq \frac{1}{\Gamma(\alpha)}\|v_n-\psi_n\|\int_0^s\int_0^r\int_0^q (q-t)^{\alpha-1}\,dtdqdr+\int_0^s\int_0^r \varepsilon_1\,dqdr$$
$$=\frac{A^{\alpha+2}}{\Gamma(\alpha+3)}\|v_n-\psi_n\|+\varepsilon_1\frac{A^2}{2}$$
$$\|u_{n+1}-\varphi_{n+1}\|\leq M\|v_n-\psi_n\|+\frac{A^2}{2}\varepsilon_1 \tag{15}$$

Similarly,

$$\|v_{n+1}-\psi_{n+1}\|=\|Tw_n-S\phi_n\|$$
$$\|v_{n+1}-\psi_{n+1}\|\leq(1-\beta_n)\|w_n-\phi_n\|+\beta_n\|Tw_n-S\phi_n\|$$
$$=(1-\beta_n)\|w_n-\phi_n\|$$
$$+\beta_n\left\|\begin{matrix}-\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^r\left(\int_0^q(q-t)^{\alpha-1}w_n(t)dt\right)dqdr+\int_0^s\int_0^r f(q,w_n(q),w_n'(q))dqdr\\-\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^r\left(\int_0^q(q-t)^{\alpha-1}\phi_n(t)dt\right)dqdr+\int_0^s\int_0^r g(q,\phi_n(q),\phi_n'(q))dqdr\end{matrix}\right\|$$
$$\leq(1-\beta_n)\|w_n-\phi_n\|$$
$$+\beta_n\frac{1}{\Gamma(\alpha)}\|w_n-\phi_n\|\int_0^s\int_0^r\int_0^q(q-t)^{\alpha-1}\,dtdqdr$$
$$+\beta_n\int_0^s\int_0^r\left\|f(q\,,w_n(q),w_n'(q))-g(q\,,\phi_n(q),\phi_n'(q))\right\|\,dqdr$$
$$\leq(1-\beta_n)\|w_n-\phi_n\|+\beta_nM\|w_n-\phi_n\|+\beta_n\varepsilon_1\frac{A^2}{2}$$
$$\|v_{n+1}-\psi_{n+1}\|\leq(1-\beta_n+\beta_nM)\|w_n-\phi_n\|+\beta_n\varepsilon_1\frac{A^2}{2} \tag{16}$$

$$\|w_n-\phi_n\|=\|Tu_n-S\varphi_n\|$$
$$\leq\left\|\begin{matrix}-\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^r\left(\int_0^q(q-t)^{\alpha-1}u_n(t)dt\right)dqdr+\int_0^s\int_0^r f(q,u_n(q),u_n'(q))dqdr\\-\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^r\left(\int_0^q(q-t)^{\alpha-1}\varphi_n(t)dt\right)dqdr+\int_0^s\int_0^r g(q,\varphi_n(q),\varphi_n'(q))dqdr\end{matrix}\right\|$$
$$\leq\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^r\int_0^q(q-t)^{\alpha-1}\,\|u_n-\varphi_n\|dtdqdr$$
$$+\int_0^s\int_0^r\left\|f(q\,,u_n(q),u_n'(q))-g(q,\varphi_n(q),\varphi_n'(q))\right\|\,dqdr$$

$$\leq \frac{1}{\Gamma(\alpha)} \|u_n - \varphi_n\| \int_0^s \int_0^r \int_0^q (q-t)^{\alpha-1}\, dtdqdr + \int_0^s \int_0^r \varepsilon_1\, dqdr$$
$$= \frac{A^{\alpha+2}}{\Gamma(\alpha+3)} \|u_n - \varphi_n\| + \varepsilon_1 \frac{A^2}{2}$$

$$\|w_n - \phi_n\| \leq M\|u_n - \varphi_n\| + \frac{A^2}{2}\varepsilon_1 \,. \tag{17}$$

Thus, if the inequality (17) is written in inequality (16) and $M \leq 1$ and $\frac{1}{2} \leq \beta_n$ is used, the following inequality is found.

$$\|v_n - \psi_n\| \leq (1 - \beta_n + \beta_n M)M\|u_n - \varphi_n\| + (1 + \beta_n M)\frac{A^2}{2}\varepsilon_1 \tag{18}$$

If this inequality (18) is written in inequality (15),

$$\|u_{n+1} - \varphi_{n+1}\| \leq M\left[(1 - \beta_n + \beta_n M)M\|u_n - \varphi_n\| + (1 + \beta_n M)\frac{A^2}{2}\varepsilon_1\right] + \frac{A^2}{2}\varepsilon_1$$
$$= M^2(1 - \beta_n + \beta_n M)\|u_n - \varphi_n\| + (1 + \beta_n M)\frac{A^2}{2}\varepsilon_1 + \frac{A^2}{2}\varepsilon_1$$
$$\leq \big(1 - \beta_n(1-M)\big)\|u_n - \varphi_n\| + (1 + \beta_n M + 1)\frac{A^2}{2}\varepsilon_1$$
$$\leq \big(1 - \beta_n(1-M)\big)\|u_n - \varphi_n\| + (1 + \beta_n + 1)\frac{A^2}{2}\varepsilon_1$$
$$\leq \big(1 - \beta_n(1-M)\big)\|u_n - \varphi_n\| + 5\beta_n\frac{A^2}{2}\varepsilon_1$$
$$= \big(1 - \beta_n(1-M)\big)\|u_n - \varphi_n\| + \beta_n(1-M)\frac{5A^2}{2(1-M)}\varepsilon_1$$

$$\|u_{n+1} - \varphi_{n+1}\| \leq [1 - \beta_n(1-M)]\|u_n - \varphi_n\| + \beta_n(1-M)\frac{5A^2\varepsilon_1}{2(1-M)} \tag{19}$$

is found. From the last inequality, we get

$$a_n = \|u_n - \varphi_n\| \,,$$
$$\mu_n = \beta_n(1-M) \in (0,1),$$
$$\gamma_n = \frac{5A^2\varepsilon_1}{2(1-M)} \geq 0.$$

Therefore, the inequality given by inequality (19) satisfies the conditions of Lemma 2.1. Then,

$$0 \leq \limsup_{n\to\infty} \|u_n - \varphi_n\| \leq \limsup_{n\to\infty} \gamma_n = \limsup_{n\to\infty} \frac{5A^2\varepsilon_1}{2(1-M)}$$

is obtained. Since $u_n \to u_*$ and $\varphi_n \to p$ as $n \to \infty$, we found

$$\|u_* - p\| \leq \frac{5A^2\varepsilon_1}{2(1-M)}. \qquad (20)$$

Conclusion

We investigated the strong convergence and data dependency of solutions to integro-differential equations using the three-step iteration method. By leveraging fixed point theory, we demonstrated the effectiveness of this iterative approach in ensuring the existence and uniqueness of solutions under specific conditions. Our findings highlight the advantages of the three-step iteration method over classical methods, particularly in terms of convergence speed and stability. Moreover, we analyzed the impact of data dependency, providing insights into the sensitivity of solutions with respect to initial conditions and parameter variations.The results obtained in this study contribute to the ongoing research in fractional differential equations and iterative methods. Future work may focus on extending these findings to more complex integro-differential systems and exploring additional iterative schemes for improved accuracy and efficiency.

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